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THE NORDHAUS-GADDUM-TYPE INEQUALITIES FOR THE NIRMALA INDICES

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Abstract. Nowadays, deducing the bounds and relations between known topological indices is an interesting tool in Chemical Graph Theory (CGT). This article investigates the mathematical properties of the recently defined Nirmala indices in terms of some graph invariants. At the outset, we establish some mathematical relations between the Nirmala indices (Nirmala index, first and second inverse Nirmala indices) and other well-established degree-based topological indices. Then, some Nordhaus-Gaddum-type inequalities for the combination of the Nirmala indices of a graph and its complement are obtained.

Keywords: Degree-based topological indices; Nirmala index; First inverse Nirmala index; Second inverse Nirmala index, Nordhaus-Gaddum-type inequalities.

AMS Subject Classification: 05C07, 05C09, 05C10, 05C90, 05C92

1. Introduction

In CGT, molecular structure of a chemical compound is interpreted as a graph in which atoms and bonds of the molecular structure are correlated to the vertices and edges of the graph, respectively. Here, molecular structures are mathematically analyzed through theoretical and computational graphical techniques. A topological index is a mathematical parameter of a molecular graph that correlates its associated physical characteristics, chemical properties and biological activity. Descriptors based on molecular graphs have been authenticated as a subsidiary in numerous chemical areas. There are many such molecular descriptors available at the present time, but very few of them have been found accessible for their feasible application.

Let $H = (V(H), E(H))$ denote a simple and connected graph where $V(H)$ and $E(H)$ represent the vertex and edge sets, respectively. For a vertex $s \in V(H)$, the degree $d_H(s)$ is the total number of edges incident to s. Also, let δ_1 and Δ_1 be the minimum and maximum degrees of H, respectively. Now, let $\overline{H}$ be the complement graph of the graph H with δ_2 and Δ_2 as its minimum and maximum degrees, respectively [21]. Furthermore,

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if we take $|E(H)| = m$ and $|V(H)| = n$ then $|V(\overline{H})| = n$ and $|E(\overline{H})| = \binom{n}{2} - m$. Also, we have the following relations of the graph H and $\overline{H}$ as follows:

$$\delta_2 = n - 1 - \Delta_1,\ \Delta_2 = n - 1 - \delta_1 \text{ and } d_H(s) + d_{\overline{H}}(s) = n - 1.$$

We now briefly discuss some notable degree-based topological indices employed in the later sections of this article. During the year 1972, I. Gutman and N. Trinajstić proposed the idea of the Zagreb indices of a graph H. The mathematical definition of the first Zagreb index ($M_1(H)$) is defined as follows:

$$M_1(H) = \sum_{s \in V(H)} d_H^2(s) \ = \sum_{st \in E(H)} (d_H(s) + d_H(t)), \tag{1}$$

whereas the second Zagreb index ($M_2(H)$) is described as

$$M_2(H) = \sum_{st \in E(H)} d_H(s) \cdot d_H(t). \tag{2}$$

In 2015, forgotten index was introduced by B. Furtula and I. Gutman in [6]. This index is a generalization of first Zagreb index and is defined as follows:

$$F(H) = \sum_{s \in V(H)} d_H^3(s) \ = \sum_{st \in E(H)} (d_H^2(s) + d_H^2(t)). \tag{3}$$

The chemical applicability of the forgotten index and its relations with the first Zagreb index and the second Zagreb index were also reported in [6, 23].

Randić index [17] is an important degree-based topological index utilized extensively in chemistry and pharmacology. In 1975, M. Randić proposed this index and defined it as follows:

$$R(H) = \sum_{st \in E(H)} \frac{1}{\sqrt{d_H(s) \cdot d_H(t)}}. \tag{4}$$

A variety of advancements and modifications have been introduced to enhance its potential power [14]. The reciprocal Randić index is one such modification that is defined as

$$RR(H) = \sum_{st \in E(H)} \sqrt{d_H(s) \cdot d_H(t)}. \tag{5}$$

The symmetric division (deg) index and inverse sum (indeg) index draw the attention of researchers because of their feasible prediction potential [1, 5, 19]. They were proposed more than 10 years ago [19] and defined as

$$SDD(H) = \sum_{st \in E(H)} \left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)} \right) \tag{6}$$

and

$$ISI(H) = \sum_{st \in E(H)} \frac{d_H(s) d_H(t)}{d_H(s) + d_H(t)}. \tag{7}$$

Recently, I. Gutman introduced the Sombor index in [7], which is a novel degree-dependent topological index. It is mathematically defined as

$$SO(H) = \sum_{st \in E(H)} \sqrt{d_H^2(s) + d_H^2(t)}. \tag{8}$$

Several articles on the Sombor index have been reported for its chemical applicability and relations with other degree-based topological indices [8, 18, 20].

Being inspired by the definition of the Sombor index, V.R. Kulli proposed the Nirmala index [9] of a graph H and mathematically defined it as follows:

$$N(H) = \sum_{st \in E(H)} \sqrt{d_H(s) + d_H(t)}. \tag{9}$$

Further, Kulli and Gutman introduced the first inverse Nirmala index (denoted as $IN_1(H)$) and second inverse Nirmala index (denoted as $IN_2(H)$) [11] of a molecular graph H and defined them as follows:

$$IN_1(H) = \sum_{st \in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} = \sum_{st \in E(H)} \sqrt{\frac{d_H(s) + d_H(t)}{d_H(s)d_H(t)}}, \tag{10}$$

$$IN_2(H) = \sum_{st \in E(H)} \frac{1}{\sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}}} = \sum_{st \in E(H)} \sqrt{\frac{d_H(s)d_H(t)}{d_H(s) + d_H(t)}}. \tag{11}$$

Currently, the M-polynomial-based derivation formulas of the Nirmala indices and its generalized version ((a, b)-Nirmala index) are proposed in [2, 3]. The computation of Nirmala indices for different dendrimers, hex-derived networks and standard graphs are reported in the articles [2, 3, 16]. A QSPR (quantitative-structure property relationship) study of the Nirmala indices to predict the physico-chemical properties of COVID-19 antiviral drugs was performed in [4] Computation of The Nirmala indices and associated entropy measures was performed in [12] for silicon carbide network.

Very recently, the structure sensitivity, abruptness and chemical applicability of some novel degree-based topological indices were tested in the article [13]. Here, a QSPR analysis was performed over the physico-chemical properties of octane isomers to test the predictive potential of the degree-based topological indices. The first inverse Nirmala index (IN_1) predicts the critical temperature (CT), molar refraction (MR), molar volume (MV) and standard heat of formation (DHFORM) with correlation-coefficients (R) -0.7378, 0.9824, 0.9503 and 0.7642, respectively of the isomers. However, the second Nirmala index (IN_2) forecasts the total surface area (TSA) of octane isomers with R-value -0.9369. The obtained results in [13] motivate us to investigate the mathematical bounds and relations of the Nirmala indices with other standard degree-based topological indices. Keeping in mind the above-discussed applicability, all three Nirmala indices can be good candidates for future experimentation in quantitative-structure property and activity relationship analysis to predict the physico-chemical properties of different molecular compounds. Also, the researchers may apply the Nirmala energy and Nirmala matrices associated with the Nirmala indices for QSPR/QSAR investigation as future work.

The main focus of this research article is to determine mathematical relations of the Nirmala indices with some well-known degree-based topological indices by involving some graph invariants. The methodology for the construction of the remaining article is discussed below. Section 2 commences with the bounds of the first and second inverse Nirmala indices of graph H in terms of its size m, maximum degree Δ_1 and minimum degree δ_1. Further, the mathematical inequalities among the Nirmala indices and some essential relations for each of the three Nirmala indices with the above-introduced notable degree-based topological indices are determined. Additionally, the Nordhaus-Gaddum-type inequalities for the combination of the Nirmala indices of a graph and its complement are established in Section 3 by using the obtained results of Section 2. Finally, we conclude in Section 4.

2. Bounds on the Nirmala Indices for General Graphs

Here, we introduce the bounds for the Nirmala indices of graph H in terms of size m, maximum degree Δ_1 and minimum degree δ_1. In addition, we present some mathematical relations among the Nirmala indices and some more well-known degree-based topological indices to deduce the Nordhaus-Gaddum-type results.

2.1. Bounds Involving Size, Maximum and Minimum Degrees.

Theorem 2.1 ([10]). *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

$$\sqrt{2\delta_1} \cdot m \leq N(H) \leq \sqrt{2\Delta_1} \cdot m,$$

with equality if and only if the graph is regular.

Proposition 2.1. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

(i) $\frac{\sqrt{2\delta_1}}{\Delta_1} \cdot m \leq IN_1(H) \leq \frac{\sqrt{2\Delta_1}}{\delta_1} \cdot m,$
(ii) $\frac{\delta_1}{\sqrt{2\Delta_1}} \cdot m \leq IN_2(H) \leq \frac{\Delta_1}{\sqrt{2\delta_1}} \cdot m.$

Moreover, the equalities hold if and only if the graph is regular.

The proof of the above theorem is straight from the definitions of the first inverse Nirmala index ($IN_1(H)$) and the second inverse Nirmala index ($IN_2(H)$), and left as an exercise to the interested readers.

2.2. Bounds Among the Nirmala Indices.

Theorem 2.2. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then, the Nirmala indices of the graph H hold the following relationship among each other*

$$\frac{2\delta_1}{\Delta_1} \cdot IN_2(H) \leq \delta_1 \cdot IN_1(H) \leq N(H) \leq \Delta_1 \cdot IN_1(H) \leq \frac{2\Delta_1}{\delta_1} \cdot IN_2(H).$$

Moreover, the equalities hold if and only if the graph is regular.

Proof. From Equations 9 and 10, we obtain

$$N(H) = \sum_{st \in E(H)} \sqrt{d_H(s) + d_H(t)} = \sum_{st \in E(H)} \sqrt{d_H(s) d_H(t)} \cdot \frac{\sqrt{d_H(s) + d_H(t)}}{\sqrt{d_H(s) d_H(t)}}$$
$$= \sum_{st \in E(H)} \sqrt{d_H(s) d_H(t)} \cdot \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \leq \Delta_1 \cdot IN_1(H).$$

Similarly, $N(H) \geq \delta_1 \cdot IN_1(H)$.

$$\therefore \; \delta_1 \cdot IN_1(H) \leq N(H) \leq \Delta_1 \cdot IN_1(H). \tag{12}$$

Next, from Equations 10 and 11, we have

$$IN_1(H) = \sum_{st \in E(H)} \sqrt{\frac{d_H(s) + d_H(t)}{d_H(s) d_H(t)}} = \sum_{st \in E(H)} \frac{d_H(s) + d_H(t)}{d_H(s) d_H(t)} \cdot \sqrt{\frac{d_H(s) d_H(t)}{d_H(s) + d_H(t)}}$$
$$= \sum_{st \in E(H)} \left(\frac{1}{d_H(s)} + \frac{1}{d_H(t)}\right) \cdot \sqrt{\frac{d_H(s) d_H(t)}{d_H(s) + d_H(t)}} \leq \frac{2}{\delta_1} \cdot IN_2(H).$$

Similarly, $IN_1(H) \geq \frac{2}{\Delta_1} \cdot IN_2(H)$.

$$\therefore \ \frac{2}{\Delta_1} \cdot IN_2(H) \leq IN_1(H) \leq \frac{2}{\delta_1} \cdot IN_2(H). \tag{13}$$

Now multiply the left and right inequalities of Equation 13 by δ_1 and Δ_1, respectively. And, then by using Equations 12 and 13 we get the required mathematical relation among the Nirmala indices.

To show the equality, let us consider H to be a r-regular graph, then $d_H(s) = r$ for every $s \in V(H)$. Also, $\delta_1 = \Delta_1 = r$. Therefore, the Nirmala index

$$N(H) = \sum_{st \in E(H)} \sqrt{d_H(s) + d_H(t)} = \sqrt{2r} \cdot m. \tag{14}$$

Now compute the following terms associated to the first inverse Nirmala index

$$\delta_1 \cdot IN_1(H) = \Delta_1 \cdot IN_1(H) = r \cdot \sum_{st \in E(H)} \sqrt{\frac{d_H(s) + d_H(t)}{d_H(s) d_H(t)}} = r \cdot \sum_{st \in E(H)} \sqrt{\frac{r + r}{r^2}} = \sqrt{2r} \cdot m. \tag{15}$$

Furthermore, examine the terms involving the second inverse Nirmala index

$$\begin{aligned} \frac{2\delta_1}{\Delta_1} \cdot IN_2(H) = \frac{2\Delta_1}{\delta_1} \cdot IN_2(H) = 2 \cdot \sum_{st \in E(H)} \sqrt{\frac{d_H(s) d_H(t)}{d_H(s) + d_H(t)}} &= 2 \cdot \sum_{st \in E(H)} \sqrt{\frac{r^2}{2r}} \\ &= \sqrt{2r} \cdot m. \end{aligned} \tag{16}$$

Combining Equations 14, 15 and 16, we obtain the required equality relationship. □

2.3. **Bounds Involving the First Zagreb Index.**

Theorem 2.3. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

(i) $\frac{1}{\sqrt{2\Delta_1}} \cdot M_1(H) \leq N(H) \leq \frac{1}{\sqrt{2\delta_1}} \cdot M_1(H)$,

(ii) $\frac{1}{\sqrt{2}\Delta_1^{3/2}} \cdot M_1(H) \leq IN_1(H) \leq \frac{1}{\sqrt{2}\delta_1^{3/2}} \cdot M_1(H)$,

(iii) $\frac{\delta_1}{2\sqrt{2}\Delta_1^{3/2}} \cdot M_1(H) \leq IN_2(H) \leq \frac{\Delta_1}{2\sqrt{2}\delta_1^{3/2}} \cdot M_1(H)$.

Moreover, equalities (left and right) hold if and only if the graph is regular.

Proof. (i) From Equations 1 and 9, we have

$$N(H) = \sum_{st \in E(H)} \sqrt{d_H(s) + d_H(t)} = \sum_{st \in E(H)} \frac{d_H(s) + d_H(t)}{\sqrt{d_H(s) + d_H(t)}} \leq \frac{1}{\sqrt{2\delta_1}} \cdot M_1(H),$$

and similarly $N(H) \geq \dfrac{1}{\sqrt{2\Delta_1}} \cdot M_1(H)$.

(ii) From Equations 1 and 10, we have

$$\begin{aligned} IN_1(H) = \sum_{st \in E(H)} \sqrt{\frac{d_H(s) + d_H(t)}{d_H(s) d_H(t)}} &= \sum_{st \in E(H)} \frac{d_H(s) + d_H(t)}{\sqrt{d_H(s) d_H(t)(d_H(s) + d_H(t))}} \\ &\leq \frac{1}{\sqrt{2}\delta_1^{3/2}} \cdot M_1(H), \end{aligned}$$

and similarly $IN_1(H) \geq \frac{1}{\sqrt{2}\Delta_1^{3/2}} \cdot M_1(H)$.

(iii) From Equations 1 and 11, we have

$$IN_2(H) = \sum_{st\in E(H)} \sqrt{\frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)}} = \sum_{st\in E(H)} \frac{\sqrt{d_H(s)d_H(t)} \cdot (d_H(s)+d_H(t))}{(d_H(s)+d_H(t))^{3/2}}$$
$$\leq \frac{\Delta_1}{2\sqrt{2}\delta_1^{3/2}} \cdot M_1(H),$$

and similarly $IN_2(H) \geq \frac{\delta_1}{2\sqrt{2}\Delta_1^{3/2}} \cdot M_1(H)$. Hence the proof.

□

2.4. **Bounds Involving the Second Zagreb Index.**

Theorem 2.4. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

(i) $\sqrt{\frac{2}{\Delta_1}} \cdot \frac{M_2(H)}{\Delta_1} \leq N(H) \leq \sqrt{\frac{2}{\delta_1}} \cdot \frac{M_2(H)}{\delta_1}$,

(ii) $\sqrt{\frac{2}{\Delta_1}} \cdot \frac{M_2(H)}{\Delta_1^2} \leq IN_1(H) \leq \sqrt{\frac{2}{\delta_1}} \cdot \frac{M_2(H)}{\delta_1^2}$,

(iii) $\frac{1}{\sqrt{2\Delta_1}} \cdot \frac{M_2(H)}{\Delta_1} \leq IN_2(H) \leq \frac{1}{\sqrt{2\delta_1}} \cdot \frac{M_2(H)}{\delta_1}$.

Moreover, equalities (left and right) holds if and only if the graph is regular.

Proof. (i) From Equations 2 and 9, we have

$$N(H) = \sum_{st\in E(H)} \sqrt{d_H(s)+d_H(t)} = \sum_{st\in E(H)} \sqrt{\frac{d_H(s)+d_H(t)}{d_H(s)d_H(t)}} \cdot \frac{d_H(s)d_H(t)}{\sqrt{d_H(s)d_H(t)}}$$
$$= \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \cdot \frac{d_H(s)d_H(t)}{\sqrt{d_H(s)d_H(t)}} \leq \sum_{st\in E(H)} \sqrt{\frac{2}{\delta_1}} \cdot \frac{d_H(s)d_H(t)}{\delta_1} = \sqrt{\frac{2}{\delta_1}} \cdot \frac{M_2(H)}{\delta_1},$$

and similarly $N(H) \geq \sqrt{\frac{2}{\Delta_1}} \cdot \frac{M_2(H)}{\Delta_1}$.

(ii) From Equations 2 and 10, we have

$$IN_1(H) = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \cdot \frac{d_H(s)d_H(t)}{d_H(s)d_H(t)}$$
$$\leq \sum_{st\in E(H)} \sqrt{\frac{2}{\delta_1}} \cdot \frac{d_H(s)d_H(t)}{\delta_1^2} = \sqrt{\frac{2}{\delta_1}} \cdot \frac{M_2(H)}{\delta_1^2},$$

and similarly $IN_1(H) \geq \sqrt{\frac{2}{\Delta_1}} \cdot \frac{M_2(H)}{\Delta_1^2}$.

(iii) From Equations 2 and 11, we have

$$IN_2(H) = \sum_{st\in E(H)} \sqrt{\frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)}} = \sum_{st\in E(H)} \frac{\sqrt{d_H(s)d_H(t)}}{\sqrt{d_H(s)+d_H(t)}} \cdot \frac{\sqrt{d_H(s)d_H(t)}}{\sqrt{d_H(s)d_H(t)}}$$
$$= \sum_{st\in E(H)} \frac{1}{\sqrt{2\delta_1}} \cdot \frac{d_H(s)d_H(t)}{\delta_1} = \frac{1}{\sqrt{2\delta_1}} \cdot \frac{M_2(H)}{\delta_1},$$

and similarly $IN_2(H) \geq \frac{1}{\sqrt{2\Delta_1}} \cdot \frac{M_2(H)}{\Delta_1}$. Hence the proof.

□

2.5. **Bounds Involving Forgotten Index.**

Theorem 2.5. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

(i) $\sqrt{\frac{\delta_1}{2}} \cdot \frac{F(H)}{\Delta_1^2} \leq N(H) \leq \sqrt{\frac{\Delta_1}{2}} \cdot \frac{F(H)}{\delta_1^2}$,

(ii) $\frac{1}{\sqrt{2\Delta_1}} \cdot \frac{F(H)}{\Delta_1^2} \leq IN_1(H) \leq \frac{1}{\sqrt{2\delta_1}} \cdot \frac{F(H)}{\delta_1^2}$,

(iii) $\sqrt{\frac{\delta_1}{2}} \cdot \frac{F(H)}{2\Delta_1^2} \leq IN_2(H) \leq \sqrt{\frac{\Delta_1}{2}} \cdot \frac{F(H)}{2\delta_1^2}$,

moreover, equalities hold if and only if the graph is regular.

Proof. (i) From Equations 3 and 9, we have

$$N(H) = \sum_{st\in E(H)} \sqrt{d_H(s) + d_H(t)} = \sum_{st\in E(H)} \frac{\sqrt{d_H(s) + d_H(t)}}{d_H^2(s) + d_H^2(t)} \cdot (d_H^2(s) + d_H^2(t))$$

$$\leq \sum_{st\in E(H)} \frac{\sqrt{2\Delta_1}}{2\delta_1^2} \cdot (d_H^2(s) + d_H^2(t)) = \sqrt{\frac{\Delta_1}{2}} \cdot \frac{F(H)}{\delta_1^2},$$

and similarly $N(H) \geq \sqrt{\frac{\delta_1}{2}} \cdot \frac{F(H)}{\delta_1^2}$.

(ii) From Equations 3 and 10, we have

$$IN_1(H) = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \cdot \frac{d_H^2(s) + d_H^2(t)}{d_H^2(s) + d_H^2(t)}$$

$$\leq \sum_{st\in E(H)} \sqrt{\frac{2}{\delta_1}} \cdot \frac{d_H^2(s) + d_H^2(t)}{2\delta_1^2} = \frac{1}{\sqrt{2\delta_1}} \cdot \frac{F(H)}{\delta_1^2},$$

and similarly $IN_1(H) \geq \frac{1}{\sqrt{2\Delta_1}} \cdot \frac{F(H)}{\Delta_1^2}$.

(iii) From Equations 3 and 11, we have

$$IN_2(H) = \sum_{st\in E(H)} \sqrt{\frac{d_H(s) d_H(t)}{d_H(s) + d_H(t)}} = \sum_{st\in E(H)} \frac{1}{\sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}}} \cdot \frac{d_H^2(s) + d_H^2(t)}{d_H^2(s) + d_H^2(t)}$$

$$\leq \sum_{st\in E(H)} \sqrt{\frac{\Delta_1}{2}} \cdot \frac{d_H^2(s) + d_H^2(t)}{2\delta_1^2} = \sqrt{\frac{\Delta_1}{2}} \cdot \frac{F(H)}{2\delta_1^2},$$

and similarly $IN_2(H) \geq \sqrt{\frac{\delta_1}{2}} \cdot \frac{F(H)}{2\Delta_1^2}$. Hence the proof.

□

2.6. **Bounds Involving the Randić Index.**

Theorem 2.6. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

(i) $\sqrt{2}\delta_1^{3/2} \cdot R(H) \leq N(H) \leq \sqrt{2}\Delta_1^{3/2} \cdot R(H)$,

(ii) $\sqrt{2\delta_1} \cdot R(H) \leq IN_1(H) \leq \sqrt{2\Delta_1} \cdot R(H)$,

(iii) $\frac{\delta_1^2}{\sqrt{2\Delta_1}} \cdot R(H) \leq IN_2(H) \leq \frac{\Delta_1^2}{\sqrt{2\delta_1}} \cdot R(H)$,
the equalities are retained if and only if the graph is regular.

Proof. (i) From Equations 4 and 9, we have

$$N(H) = \sum_{st\in E(H)} \sqrt{d_H(s)+d_H(t)} = \sum_{st\in E(H)} \sqrt{d_H(s)+d_H(t)} \cdot \frac{\sqrt{d_H(s)d_H(t)}}{\sqrt{d_H(s)d_H(t)}}$$

$$\leq \sum_{st\in E(H)} \sqrt{2\Delta_1} \cdot \frac{\Delta_1}{\sqrt{d_H(s)d_H(t)}} = \sqrt{2}\Delta_1^{3/2} \cdot R(H),$$

and similarly $N(H) \geq \sqrt{2}\delta_1^{3/2} \cdot R(H)$.

(ii) From Equations 4 and 10, we have

$$IN_1(H) = \sum_{st\in E(H)} \sqrt{\frac{d_H(s)+d_H(t)}{d_H(s)d_H(t)}} \leq \sum_{st\in E(H)} \frac{\sqrt{2\Delta_1}}{\sqrt{d_H(s)d_H(t)}} = \sqrt{2\Delta_1} \cdot R(H),$$

and similarly $IN_1(H) \geq \sqrt{2\delta_1} \cdot R(H)$.

(iii) From Equations 4 and 11, we have

$$IN_2(H) = \sum_{st\in E(H)} \sqrt{\frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)}} = \sum_{st\in E(H)} \frac{d_H(s)d_H(t)}{\sqrt{d_H(s)+d_H(t)}} \cdot \frac{1}{\sqrt{d_H(s)d_H(t)}}$$

$$\leq \sum_{st\in E(H)} \frac{\Delta_1^2}{\sqrt{2\delta_1}} \cdot \frac{1}{\sqrt{d_H(s)d_H(t)}} = \frac{\Delta_1^2}{\sqrt{2\delta_1}} \cdot R(H),$$

and similarly $IN_2(H) \geq \frac{\delta_1^2}{\sqrt{2\Delta_1}} \cdot R(H)$. Hence the proof.

□

2.7. **Bounds Involving the Reciprocal Randić Index.**

Theorem 2.7. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

(i) $\sqrt{\frac{2}{\Delta_1}} \cdot RR(H) \leq N(H) \leq \sqrt{\frac{2}{\delta_1}} \cdot RR(H)$,
(ii) $\sqrt{\frac{2}{\Delta_1}} \cdot \frac{RR(H)}{\Delta_1} \leq IN_1(H) \leq \sqrt{\frac{2}{\delta_1}} \cdot \frac{RR(H)}{\delta_1}$,
(iii) $\frac{1}{\sqrt{2\Delta_1}} \cdot RR(H) \leq IN_2(H) \leq \frac{1}{\sqrt{2\delta_1}} \cdot RR(H)$.
Moreover, equality (left and right) holds if and only if the graph is regular.

Proof. (i) From Equations 5 and 9, we have

$$N(H) = \sum_{st\in E(H)} \sqrt{d_H(s)+d_H(t)} = \sum_{st\in E(H)} \frac{\sqrt{d_H(s)+d_H(t)}}{\sqrt{d_H(s)d_H(t)}} \cdot \sqrt{d_H(s)d_H(t)}$$

$$= \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \cdot \sqrt{d_H(s)d_H(t)}$$

$$\leq \sum_{st\in E(H)} \sqrt{\frac{2}{\delta_1}} \cdot \sqrt{d_H(s)d_H(t)} = \sqrt{\frac{2}{\delta_1}} \cdot RR(H),$$

and similarly $N(H) \geq \sqrt{\frac{2}{\Delta_1}} \cdot RR(H)$.

(ii) From Equations 5 and 10, we have

$$IN_1(H) = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \cdot \frac{\sqrt{d_H(s)d_H(t)}}{\sqrt{d_H(s)d_H(t)}}$$
$$\leq \sum_{st\in E(H)} \sqrt{\frac{2}{\delta_1}} \cdot \frac{\sqrt{d_H(s)d_H(t)}}{\delta_1} = \sqrt{\frac{2}{\delta_1}} \cdot \frac{RR(H)}{\delta_1},$$

and similarly $IN_1(H) \geq \sqrt{\dfrac{2}{\Delta_1}} \cdot \dfrac{RR(H)}{\Delta_1}$.

(iii) From Equations 5 and 11, we have

$$IN_2(H) = \sum_{st\in E(H)} \sqrt{\frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)}} \leq \sum_{st\in E(H)} \frac{\sqrt{d_H(s)d_H(t)}}{\sqrt{2\delta_1}} \leq \frac{RR(H)}{\sqrt{2\delta_1}},$$

and similarly $IN_2(H) \geq \frac{RR(H)}{\sqrt{2\Delta_1}}$. Hence the proof. □

2.8. **Bounds Involving Symmetric Division (Deg) Index.**

Theorem 2.8. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

(i) $\sqrt{\frac{\delta_1}{2}} \cdot \frac{\delta_1}{\Delta_1} \cdot SDD(H) \leq N(H) \leq \sqrt{\frac{\Delta_1}{2}} \cdot \frac{\Delta_1}{\delta_1} \cdot SDD(H)$,

(ii) $\frac{\delta_1}{\Delta_1} \cdot \frac{SDD(H)}{\sqrt{2\Delta_1}} \leq IN_1(H) \leq \frac{\Delta_1}{\delta_1} \cdot \frac{SDD(H)}{\sqrt{2\delta_1}}$,

(iii) $\sqrt{\frac{\delta_1}{2}} \cdot \frac{\delta_1}{2\Delta_1} \cdot SDD(H) \leq IN_2(H) \leq \sqrt{\frac{\Delta_1}{2}} \cdot \frac{\Delta_1}{2\delta_1} \cdot SDD(H)$.

Moreover, in the above boundness, the equalities hold if and only if the graph is regular.

Proof. (i) From Equations 6 and 9, we have

$$N(H) = \sum_{st\in E(H)} \sqrt{d_H(s)+d_H(t)} = \sum_{st\in E(H)} \frac{\sqrt{d_H(s)+d_H(t)}}{\left(\frac{d_H^2(s)+d_H^2(t)}{d_H(s)\cdot d_H(t)}\right)} \cdot \left(\frac{d_H^2(s)+d_H^2(t)}{d_H(s)\cdot d_H(t)}\right)$$
$$= \sum_{st\in E(H)} \frac{\sqrt{d_H(s)+d_H(t)}}{\left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)}\right)} \cdot \left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)}\right)$$
$$\leq \sum_{st\in E(H)} \frac{\sqrt{2\Delta_1}}{\frac{2\delta_1}{\Delta_1}} \cdot \left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)}\right) = \sqrt{\frac{\Delta_1}{2}} \cdot \frac{\Delta_1}{\delta_1} \cdot SDD(H),$$

and similarly $N(H) \geq \sqrt{\dfrac{\delta_1}{2}} \cdot \dfrac{\delta_1}{\Delta_1} \cdot SDD(H)$.

(ii) From Equations 6 and 10, we have

$$IN_1(H) = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}}$$

$$= \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \cdot \frac{1}{\left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)}\right)} \cdot \left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)}\right)$$

$$\leq \sum_{st\in E(H)} \sqrt{\frac{2}{\delta_1}} \cdot \frac{\Delta_1}{2\delta_1} \cdot \left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)}\right) = \frac{\Delta_1}{\delta_1} \cdot \frac{SDD(H)}{\sqrt{2\delta_1}},$$

and similarly $IN_1(H) \geq \frac{\delta_1}{\Delta_1} \cdot \frac{SDD(H)}{\sqrt{2\Delta_1}}$.

(iii) From Equations 6 and 11, we have

$$IN_2(H) = \sum_{st\in E(H)} \frac{1}{\sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}}}$$

$$= \sum_{st\in E(H)} \frac{1}{\sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}}} \cdot \frac{1}{\left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)}\right)} \cdot \left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)}\right)$$

$$\leq \sum_{st\in E(H)} \sqrt{\frac{\Delta_1}{2}} \cdot \frac{\Delta_1}{2\delta_1} \cdot \left(\frac{d_H(s)}{d_H(t)} + \frac{d_H(t)}{d_H(s)}\right) = \sqrt{\frac{\Delta_1}{2}} \cdot \frac{\Delta_1}{2\delta_1} \cdot SDD(H),$$

and similarly $IN_2(H) \geq \sqrt{\frac{\delta_1}{2}} \cdot \frac{\delta_1}{2\Delta_1} \cdot SDD(H)$. Hence the proof.

□

2.9. **Bounds Involving Inverse Sum (Indeg) Index.**

Theorem 2.9. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively.*

(i) $\frac{2\sqrt{2\delta_1}}{\Delta_1} \cdot ISI(H) \leq N(H) \leq \frac{2\sqrt{2\Delta_1}}{\delta_1} \cdot ISI(H)$,

(ii) $\frac{2\sqrt{2}}{\Delta_1^{3/2}} \cdot ISI(H) \leq IN_1(H) \leq \frac{2\sqrt{2}}{\delta_1^{3/2}} \cdot ISI(H)$,

(iii) $\sqrt{\frac{2}{\Delta_1}} \cdot ISI(H) \leq IN_2(H) \leq \sqrt{\frac{2}{\delta_1}} \cdot ISI(H)$.

Moreover, in the above boundness, the equalities hold if and only if the graph is regular.

Proof. (i) From Equations 7 and 9, we have

$$N(H) = \sum_{st\in E(H)} \sqrt{d_H(s) + d_H(t)}$$

$$= \sum_{st\in E(H)} \sqrt{d_H(s) + d_H(t)} \cdot \frac{d_H(s) + d_H(t)}{d_H(s)d_H(t)} \cdot \frac{d_H(s)d_H(t)}{d_H(s) + d_H(t)}$$

$$= \sum_{st\in E(H)} \frac{d_H(s)d_H(t)}{d_H(s) + d_H(t)} \cdot \left(\frac{1}{d_H(s)} + \frac{1}{d_H(t)}\right) \cdot \sqrt{d_H(s) + d_H(t)} \leq \frac{2\sqrt{2\Delta_1}}{\delta_1} \cdot ISI(H),$$

and similarly $N(H) \geq \frac{2\sqrt{2\delta_1}}{\Delta_1} \cdot ISI(H)$.

(ii) From Equations 7 and 10, we have

$$IN_1(H) = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}}$$

$$= \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \cdot \frac{d_H(s)+d_H(t)}{d_H(s)d_H(t)} \cdot \frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)}$$
$$= \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \cdot \Big(\frac{1}{d_H(s)} + \frac{1}{d_H(t)}\Big) \cdot \frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)}$$
$$\leq \frac{2\sqrt{2}}{\delta_1^{3/2}} \cdot ISI(H),$$

and similarly $IN_1(H) \geq \frac{2\sqrt{2}}{\Delta_1^{3/2}} \cdot ISI(H)$.

(iii) From Equations 7 and 11, we have

$$IN_2(H) = \sum_{st\in E(H)} \sqrt{\frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)}}$$
$$= \sum_{st\in E(H)} \sqrt{\frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)}} \cdot \frac{\sqrt{d_H(s)+d_H(t)}}{\sqrt{d_H(s)d_H(t)}} \cdot \frac{\sqrt{d_H(s)d_H(t)}}{\sqrt{d_H(s)+d_H(t)}}$$
$$= \sum_{st\in E(H)} \frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)} \cdot \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \leq \sqrt{\frac{2}{\delta_1}} \cdot ISI(H),$$

and similarly $IN_2(H) \geq \sqrt{\frac{2}{\Delta_1}} \cdot ISI(H)$. Hence the proof. □

2.10. **Bounds Involving the Sombor Index.**

Theorem 2.10 ([10]). *Let H be the connected graph of order n and size m with maximum and minimum degrees Δ_1 and δ_1, respectively. Then $\frac{SO(H)}{\sqrt{\Delta_1}} \leq N(H) \leq \frac{SO(H)}{\sqrt{\delta_1}}$. Moreover, equality (left and right) holds if and only if the graph is regular.*

Theorem 2.11. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

(i) $\frac{1}{\Delta_1^{3/2}} \cdot SO(H) \leq IN_1(H) \leq \frac{1}{\delta_1^{3/2}} \cdot SO(H)$,

(ii) $\frac{\sqrt{\delta_1}}{2\Delta_1} \cdot SO(H) \leq IN_2(H) \leq \frac{\sqrt{\Delta_1}}{2\delta_1} \cdot SO(H)$.

Moreover, the equalities are attained if and only if the graph is regular.

Proof. (i) From Equations 8 and 10, we have

$$IN_1(H) = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} = \sum_{st\in E(H)} \sqrt{\frac{1}{d_H(s)} + \frac{1}{d_H(t)}} \cdot \frac{\sqrt{d_H^2(s)+d_H^2(t)}}{\sqrt{d_H^2(s)+d_H^2(t)}}$$
$$\leq \sum_{st\in E(H)} \sqrt{\frac{2}{\delta_1}} \cdot \frac{\sqrt{d_H^2(s)+d_H^2(t)}}{\sqrt{{\delta_1}^2+{\delta_1}^2}} = \frac{1}{\delta_1^{3/2}} \cdot SO(H),$$

and similarly $IN_1(H) \geq \dfrac{1}{\Delta_1^{3/2}} \cdot SO(H)$.

(ii) From Equations 8 and 11, we have

$$IN_2(H) = \sum_{st \in E(H)} \sqrt{\frac{d_H(s)d_H(t)}{d_H(s)+d_H(t)}} = \sum_{st \in E(H)} \frac{1}{\sqrt{\frac{1}{d_H(s)}+\frac{1}{d_H(t)}}} \cdot \frac{\sqrt{d_H^2(s)+d_H^2(t)}}{\sqrt{d_H^2(s)+d_H^2(t)}}$$

$$\leq \sum_{st \in E(H)} \frac{1}{\sqrt{\frac{2}{\Delta_1}}} \cdot \frac{\sqrt{d_H^2(s)+d_H^2(t)}}{\sqrt{{\delta_1}^2+{\delta_1}^2}} = \frac{\sqrt{\Delta_1}}{2\delta_1} \cdot SO(H),$$

and similarly $IN_2(H) \geq \frac{\sqrt{\delta_1}}{2\Delta_1} \cdot SO(H)$. Hence the proof. □

3. Nordhaus-Gaddum-type Inequalities for the Nirmala Indices

Nordhaus-Gaddum results have been often reported in the field of CGT to investigate graph invariants. In 1956, E.A. Nordhaus and J.W. Gaddum investigated the bounds of the chromatic number of the graph and its complement [15]. Li Zhang and Baoyindureng Wu gave the Nordhaus-Gaddum type inequalities [22] for the Zagreb index, general Randić index and the Wiener index in 2005, inspired by the bounds of chromatic number. Recently, for the Sombor index, Nordhus-Gaddum type bounds have been reported in [20]. Motivated by these results, we establish some Nordhaus-Gaddum-type results for the combination of the Nirmala indices of graph H and its complement with the help of previously derived results.

Proposition 3.1. *Let us consider a connected graph H with order n and size m having maximum and minimum degrees Δ_1 and δ_1, respectively. Then*

(i) $m \cdot \sqrt{2\delta_1} + \left[\binom{n}{2} - m\right] \cdot \sqrt{2(n-1-\Delta_1)} \leq N(H) + N(\overline{H}) \leq m \cdot \sqrt{2\Delta_1}$
$+ \left[\binom{n}{2} - m\right] \cdot \sqrt{2(n-1-\delta_1)}$

(ii) $m \cdot \frac{\sqrt{2\delta_1}}{\Delta_1} + \left[\binom{n}{2} - m\right] \cdot \frac{\sqrt{2(n-1-\Delta_1)}}{(n-1-\delta_1)} \leq IN_1(H) + IN_1(\overline{H}) \leq m \cdot \frac{\sqrt{2\Delta_1}}{\delta_1}$
$+ \left[\binom{n}{2} - m\right] \cdot \frac{\sqrt{2(n-1-\delta_1)}}{(n-1-\Delta_1)}$,

(iii) $m \cdot \frac{\delta}{\sqrt{2\Delta_1}} + \left[\binom{n}{2} - m\right] \cdot \frac{(n-1-\delta_1)}{\sqrt{2(n-1-\Delta_1)}} \leq IN_2(H) + IN_2(\overline{H}) \leq m \cdot \frac{\Delta}{\sqrt{2\delta_1}}$
$+ \left[\binom{n}{2} - m\right] \cdot \frac{(n-1-\delta_1)}{\sqrt{2(n-1-\Delta_1)}}$.

Proof. Since $|E(H)| = m$ and $|V(H)| = n$ then $|V(\overline{H})| = n$, and $|E(\overline{H})| = \binom{n}{2} - m$. Accordingly, the minimum and maximum degrees of $\overline{H}$ are given by $\delta_2 = n-1-\Delta_1$ and $\Delta_2 = n-1-\delta_1$, respectively. Therefore, by employing Theorem 2.1, the result (i) is evident. Whereas, with the help of Proposition 2.1 for the graph H and its complement $\overline{H}$, one can easily show the other two results. □

Theorem 3.1 ([22])**.** *Let H be a connected graph of order n and size m. Then*

$$\frac{n(n-1)^2}{2} \leq M_1(H) + M_1(\overline{H}) \leq n(n-1)^2.$$

Theorem 3.2. *Let us consider a connected graph H with order n and size m. Then*

(i) $\frac{n(n-1)^2}{2\sqrt{2} \cdot \max\{\sqrt{\Delta_1}, \sqrt{\Delta_2}\}} \leq N(H) + N(\overline{H}) \leq \frac{n(n-1)^2 - 2n\delta_1\delta_2}{\sqrt{2} \cdot \min\{\sqrt{\delta_1}, \sqrt{\delta_2}\}}$,

(ii) $\frac{n(n-1)^2}{2\sqrt{2} \cdot \max\{\Delta_1^{3/2}, \Delta_2^{3/2}\}} \leq IN_1(H) + IN_1(\overline{H}) \leq \frac{n(n-1)^2 - 2n\delta_1\delta_2}{\sqrt{2} \cdot \max\{\delta_1^{3/2}, \delta_2^{3/2}\}}$,

(iii) $\min\left\{\frac{\delta_1}{\Delta_1^{3/2}},\frac{\delta_2}{\Delta_2^{3/2}}\right\}\left(\frac{n(n-1)^2}{4\sqrt{2}}\right)\leq IN_2(H)+IN_2(\overline{H})\leq \min\left\{\frac{\Delta_1}{\delta_1^{3/2}},\frac{\Delta_2}{\delta_2^{3/2}}\right\}\cdot\left(\frac{n(n-1)^2-2n\delta_1\delta_2}{2\sqrt{2}}\right)$.

Proof. From Theorems 2.3, 3.1 and Equation 1, we have

$$
\begin{aligned}
\text{(i) } N(H)+N(\overline{H}) &\leq \frac{M_1(H)}{\sqrt{2\delta_1}}+\frac{M_1(\overline{H})}{\sqrt{2\delta_2}}\leq \frac{M_1(H)+M_1(\overline{H})}{\sqrt{2}\cdot\min\{\sqrt{\delta_1},\sqrt{\delta_2}\}}\\
&\leq \frac{\left[\sum_{s\in V(H)} d_H^2(s)+\sum_{s\in V(\overline{H})} d_{\overline{H}}^2(s)\right]}{\sqrt{2}\cdot\min\{\sqrt{\delta_1},\sqrt{\delta_2}\}}=\frac{\left[\sum_{s\in V(H)}\left(d_H^2(s)+d_{\overline{H}}^2(s)\right)\right]}{\sqrt{2}\cdot\min\{\sqrt{\delta_1},\sqrt{\delta_2}\}}\\
&= \frac{\sum_{s\in V(H)}\left[\left(d_H(s)+d_{\overline{H}}(s)\right)^2-2d_H(s)d_{\overline{H}}(s)\right]}{\sqrt{2}\cdot\min\{\sqrt{\delta_1},\sqrt{\delta_2}\}}\leq \frac{n(n-1)^2-2n\delta_1\delta_2}{\sqrt{2}\cdot\min\{\sqrt{\delta_1},\sqrt{\delta_2}\}},\\
\text{and } N(H)+N(\overline{H}) &\geq \frac{M_1(H)}{\sqrt{2\Delta_1}}+\frac{M_1(\overline{H})}{\sqrt{2\Delta_2}}\geq \frac{M_1(H)+M_1(\overline{H})}{\sqrt{2}\cdot\max\{\sqrt{\Delta_1},\sqrt{\Delta_2}\}}\\
&\geq \frac{n(n-1)^2}{2\sqrt{2}\cdot\max\{\sqrt{\Delta_1},\sqrt{\Delta_2}\}},
\end{aligned}
$$

$$
\begin{aligned}
\text{(ii) } IN_1(H)+IN_1(\overline{H}) &\leq \frac{M_1(H)}{\sqrt{2}\delta_1^{3/2}}+\frac{M_1(\overline{H})}{\sqrt{2}\delta_2^{3/2}}\leq \frac{M_1(H)+M_1(\overline{H})}{\sqrt{2}\cdot\min\{\delta_1^{3/2},\delta_2^{3/2}\}}\\
&\leq \frac{n(n-1)^2-2n\delta_1\delta_2}{\sqrt{2}\cdot\min\{\delta_1^{3/2},\delta_2^{3/2}\}},\\
\text{and } IN_1(H)+IN_1(\overline{H}) &\geq \frac{M_1(H)}{\sqrt{2}\Delta_1^{3/2}}+\frac{M_1(\overline{H})}{\sqrt{2}\Delta_2^{3/2}}\geq \frac{M_1(H)+M_1(\overline{H})}{\sqrt{2}\cdot\max\{\Delta_1^{3/2},\Delta_2^{3/2}\}}\\
&\geq \frac{n(n-1)^2}{2\sqrt{2}\cdot\max\{\Delta_1^{3/2},\Delta_2^{3/2}\}}.
\end{aligned}
$$

$$
\begin{aligned}
\text{(iii) } IN_2(H)+IN_2(\overline{H}) &\leq \frac{\Delta_1 M_1(H)}{2\sqrt{2}{\delta_1}^{3/2}}+\frac{\Delta_2 M_1(\overline{H})}{2\sqrt{2}\delta_2^{3/2}}\\
&\leq \frac{1}{2\sqrt{2}}\max\left\{\frac{\Delta_1}{\delta_1^{3/2}},\frac{\Delta_2}{\delta_2^{3/2}}\right\}\cdot\left[M_1(H)+M_1(\overline{H})\right]\\
&\leq \frac{1}{2\sqrt{2}}\max\left\{\frac{\Delta_1}{\delta_1^{3/2}},\frac{\Delta_2}{\delta_2^{3/2}}\right\}\cdot\left(n(n-1)^2-2n\delta_1\delta_2\right),\\
\text{and } IN_2(H)+IN_2(\overline{H}) &\geq \frac{\delta_1 M_1(H)}{2\sqrt{2}{\Delta_1}^{3/2}}+\frac{\delta_2 M_1(\overline{H})}{2\sqrt{2}\Delta_2^{3/2}}\\
&\geq \frac{1}{2\sqrt{2}}\min\left\{\frac{\delta_1}{\Delta_1^{3/2}},\frac{\delta_2}{\Delta_2^{3/2}}\right\}\cdot\left[M_1(H)+M_1(\overline{H})\right]\\
&\geq \frac{1}{4\sqrt{2}}\min\left\{\frac{\delta_1}{\Delta_1^{3/2}},\frac{\delta_2}{\Delta_2^{3/2}}\right\}\cdot\left(n(n-1)^2\right).
\end{aligned}
$$

□

Theorem 3.3. *Let H be a connected graph of order n and size m. Then* *(i)* $\frac{1}{\sqrt{2}} \cdot \min\left\{\frac{\sqrt{\delta_1}}{\Delta_1^2}, \frac{\sqrt{\delta_2}}{\Delta_2^2}\right\} \cdot (n(n-1)^3 - 3n(n-1)\Delta_1\Delta_2) \leq N(H) + N(\overline{H}) \leq \frac{1}{\sqrt{2}}$ $\cdot \max\left\{\frac{\sqrt{\Delta_1}}{\delta_1^2}, \frac{\sqrt{\Delta_2}}{\delta_2^2}\right\} \cdot (n(n-1)^3 - 3n(n-1)\delta_1\delta_2)$,

(ii) $\frac{1}{\sqrt{2}} \cdot \frac{(n(n-1)^3 - 3n(n-1)\Delta_1\Delta_2)}{\max\{\Delta_1^{5/2}, \Delta_2^{5/2}\}} \leq IN_1(H) + IN_1(\overline{H}) \leq \frac{1}{\sqrt{2}} \cdot \frac{(n(n-1)^3 - 3n(n-1)\delta_1\delta_2)}{\min\{\delta_1^{5/2}, \delta_2^{5/2}\}}$,

(iii) $\frac{1}{2\sqrt{2}} \cdot \min\left\{\frac{\sqrt{\delta_1}}{\Delta_1^2}, \frac{\sqrt{\delta_2}}{\Delta_2^2}\right\} \cdot (n(n-1)^3 - 3n(n-1)\Delta_1\Delta_2) \leq IN_2(H) + IN_2(\overline{H})$ $\leq \frac{1}{2\sqrt{2}} \cdot \max\left\{\frac{\sqrt{\Delta_1}}{\delta_1^2}, \frac{\sqrt{\Delta_2}}{\delta_2^2}\right\} \cdot (n(n-1)^3 - 3n(n-1)\delta_1\delta_2)$.

Proof. From Theorem 2.5 and Equation 3, we have

$$
\begin{aligned}
\text{(i) } N(H) + N(\overline{H}) &\leq \sqrt{\frac{\Delta_1}{2}} \cdot \frac{F(H)}{\delta_1^2} + \sqrt{\frac{\Delta_2}{2}} \cdot \frac{F(\overline{H})}{\delta_2^2} \\
&\leq \frac{1}{\sqrt{2}} \cdot \max\left\{\frac{\sqrt{\Delta_1}}{\delta_1^2}, \frac{\sqrt{\Delta_2}}{\delta_2^2}\right\} \cdot \left[F(H) + F(\overline{H})\right] \\
&= \frac{1}{\sqrt{2}} \cdot \max\left\{\frac{\sqrt{\Delta_1}}{\delta_1^2}, \frac{\sqrt{\Delta_2}}{\delta_2^2}\right\} \cdot \Big[\sum_{s \in V(H)} d_H^3(s) + \sum_{s \in V(\overline{H})} d_{\overline{H}}^3(s)\Big] \\
&= \frac{1}{\sqrt{2}} \cdot \max\left\{\frac{\sqrt{\Delta_1}}{\delta_1^2}, \frac{\sqrt{\Delta_2}}{\delta_2^2}\right\} \cdot \Big[\sum_{s \in V(H)} \left(d_H^3(s) + d_{\overline{H}}^3(s)\right)\Big] \\
&= \frac{1}{\sqrt{2}} \cdot \max\left\{\frac{\sqrt{\Delta_1}}{\delta_1^2}, \frac{\sqrt{\Delta_2}}{\delta_2^2}\right\} \\
&\quad \cdot \sum_{s \in V(H)} \left[\left(d_H(s) + d_{\overline{H}}(s)\right)^3 - 3d_H(s)d_{\overline{H}}(s)(d_H(s) + d_{\overline{H}}(s))\right] \\
&\leq \frac{1}{\sqrt{2}} \cdot \max\left\{\frac{\sqrt{\Delta_1}}{\delta_1^2}, \frac{\sqrt{\Delta_2}}{\delta_2^2}\right\} \cdot (n(n-1)^3 - 3n(n-1)\delta_1\delta_2),
\end{aligned}
$$

$$
\begin{aligned}
\text{and } N(H) + N(\overline{H}) &\geq \sqrt{\frac{\delta_1}{2}} \cdot \frac{F(H)}{\Delta_1^2} + \sqrt{\frac{\delta_2}{2}} \cdot \frac{F(\overline{H})}{\Delta_2^2} \\
&\geq \frac{1}{\sqrt{2}} \cdot \min\left\{\frac{\sqrt{\delta_1}}{\Delta_1^2}, \frac{\sqrt{\delta_2}}{\Delta_2^2}\right\} \cdot \left[F(H) + F(\overline{H})\right] \\
&\geq \frac{1}{\sqrt{2}} \cdot \min\left\{\frac{\sqrt{\delta_1}}{\Delta_1^2}, \frac{\sqrt{\delta_2}}{\Delta_2^2}\right\} \cdot (n(n-1)^3 - 3n(n-1)\Delta_1\Delta_2).
\end{aligned}
$$

$$
\begin{aligned}
\text{(ii) } IN_1(H) + IN_1(\overline{H}) &\leq \frac{F(H)}{\sqrt{2}\delta_1^{5/2}} + \frac{F(\overline{H})}{\sqrt{2}\delta_2^{5/2}} \leq \frac{F(H) + F(\overline{H})}{\sqrt{2} \cdot \min\{\delta_1^{5/2}, \delta_2^{5/2}\}} \\
&\leq \frac{n(n-1)^3 - 3n(n-1)\delta_1\delta_2}{\sqrt{2} \cdot \min\{\delta_1^{5/2}, \delta_2^{5/2}\}},
\end{aligned}
$$

$$
\begin{aligned}
\text{and } IN_1(H) + IN_1(\overline{H}) &\geq \frac{F(H)}{\sqrt{2}\Delta_1^{5/2}} + \frac{F(\overline{H})}{\sqrt{2}\Delta_2^{5/2}} \geq \frac{F(H) + F(\overline{H})}{\sqrt{2} \cdot \max\{\Delta_1^{5/2}, \Delta_2^{5/2}\}} \\
&\geq \frac{n(n-1)^3 - 3n(n-1)\Delta_1\Delta_2}{\sqrt{2} \cdot \max\{\Delta_1^{5/2}, \Delta_2^{5/2}\}},
\end{aligned}
$$

$$
\text{(iii) } IN_2(H) + IN_2(\overline{H}) \leq \sqrt{\frac{\Delta_1}{2}} \cdot \frac{F(H)}{2\delta_1^2} + \sqrt{\frac{\Delta_2}{2}} \cdot \frac{F(\overline{H})}{2\delta_2^2}
$$

$$\le \frac{1}{2\sqrt{2}} \cdot \max\left\{\frac{\sqrt{\Delta_1}}{\delta_1^2}, \frac{\sqrt{\Delta_2}}{\delta_2^2}\right\} \cdot \big[F(H) + F(\overline{H})\big]$$

$$\le \frac{1}{2\sqrt{2}} \cdot \max\left\{\frac{\sqrt{\Delta_1}}{\delta_1^2}, \frac{\sqrt{\Delta_2}}{\delta_2^2}\right\} \cdot \big(n(n-1)^3 - 3n(n-1)\delta_1\delta_2\big),$$

$$\text{and } IN_2(H) + IN_2(\overline{H}) \ge \sqrt{\frac{\delta_1}{2}} \cdot \frac{F(H)}{2\Delta_1^2} + \sqrt{\frac{\delta_2}{2}} \cdot \frac{F(\overline{H})}{2\Delta_2^2}$$

$$\ge \frac{1}{2\sqrt{2}} \cdot \min\left\{\frac{\sqrt{\delta_1}}{\Delta_1^2}, \frac{\sqrt{\delta_2}}{\Delta_2^2}\right\} \cdot \big[F(H) + F(\overline{H})\big]$$

$$\ge \frac{1}{2\sqrt{2}} \cdot \min\left\{\frac{\sqrt{\delta_1}}{\Delta_1^2}, \frac{\sqrt{\delta_2}}{\Delta_2^2}\right\} \cdot \big(n(n-1)^3 - 3n(n-1)\Delta_1\Delta_2\big).$$

□

Theorem 3.4 ([20]). *Let H be a connected graph of order n and size m. Then*

$$m\delta_1\sqrt{2} + \left[\binom{n}{2} - m\right](n-1-\Delta_1)\sqrt{2} \le SO(H) + SO(\overline{H}) \le m\Delta_1\sqrt{2} + \left[\binom{n}{2} - m\right](n-1-\delta_1)\sqrt{2}.$$

Theorem 3.5. *Let us consider a connected graph H with order n and size m. Then*

(i) $\frac{1}{\max\{\sqrt{\Delta_1}, \sqrt{\Delta_2}\}} \cdot \Big(m\delta_1\sqrt{2} + \Big[\binom{n}{2} - m\Big](n-1-\Delta_1)\sqrt{2}\Big) \le N(H) + N(\overline{H}) \le \frac{1}{\max\{\sqrt{\delta_1}, \sqrt{\delta_2}\}} \cdot \Big(m\Delta_1\sqrt{2} + \Big[\binom{n}{2} - m\Big](n-1-\delta_1)\sqrt{2}\Big)$,

(ii) $\frac{1}{\max\{\Delta_1^{3/2}, \Delta_2^{3/2}\}} \cdot \Big(m\delta_1\sqrt{2} + \Big[\binom{n}{2} - m\Big](n-1-\Delta_1)\sqrt{2}\Big) \le IN_1(H) + IN_1(\overline{H}) \le \frac{1}{\max\{\delta_1^{3/2}, \delta_2^{3/2}\}} \cdot \Big(m\Delta_1\sqrt{2} + \Big[\binom{n}{2} - m\Big](n-1-\delta_1)\sqrt{2}\Big)$,

(iii) $\min\left\{\frac{\sqrt{\delta_1}}{2\Delta_1}, \frac{\sqrt{\delta_2}}{2\Delta_2}\right\} \cdot \Big(m\delta_1\sqrt{2} + \Big[\binom{n}{2} - m\Big](n-1-\Delta_1)\sqrt{2}\Big) \le IN_2(H) + IN_2(\overline{H}) \le \max\left\{\frac{\sqrt{\Delta_1}}{2\delta_1}, \frac{\sqrt{\Delta_2}}{2\delta_2}\right\} \cdot \Big(m\Delta_1\sqrt{2} + \Big[\binom{n}{2} - m\Big](n-1-\delta_1)\sqrt{2}\Big)$.

Proof. From Theorems 2.10, 2.11 and 3.4, the proofs of the above upper and lower bounds for the Nirmala indices are immediate as proved in Theorem 3.2. □

Theorem 3.6 ([22]). *Let H be a connected graph of order n and size m. Then*

$$\frac{n(n-1)}{2(n-2)} \le R(H) + R(\overline{H}) \le n.$$

Theorem 3.7. *Let us consider a connected graph H with order n and size m. Then*

(i) $\sqrt{2}\min\left\{\delta_1^{3/2}, \delta_2^{3/2}\right\} \cdot \frac{n(n-1)}{2(n-2)} \le N(H) + N(\overline{H}) \le \sqrt{2}\min\left\{\Delta_1^{3/2}, \Delta_2^{3/2}\right\} \cdot n$,

(ii) $\min\left\{\sqrt{2\delta_1}, \sqrt{2\delta_2}\right\} \cdot \frac{n(n-1)}{2(n-2)} \le IN_1(H) + IN_1(\overline{H}) \le \max\left\{\sqrt{2\Delta_1}, \sqrt{2\Delta_2}\right\} \cdot n$,

(iii) $\min\left\{\frac{\delta_1^2}{\sqrt{2\Delta_1}}, \frac{\delta_2^2}{\sqrt{2\Delta_2}}\right\} \cdot \frac{n(n-1)}{2(n-2)} \le IN_2(H) + IN_2(\overline{H}) \le \max\left\{\frac{\Delta_1^2}{\sqrt{2\delta_1}}, \frac{\Delta_2^2}{\sqrt{2\delta_2}}\right\} \cdot n$.

Proof. The proof of the above upper bound and lower bound for the Nirmala indices can be proved similar to Theorem 3.2 by using Theorems 2.6 and 3.6. □

4. Conclusion

In this current study, in the first place, some bounds of the Nirmala indices (Nirmala index, first inverse Nirmala index and second inverse Nirmala index) in terms of some graph invariants were substantiated and mathematical inequalities among the Nirmala indices were established. Later, the bounds of the Nirmala indices in terms of several standard degree-based topological indices were proposed. Furthermore, Nordhaus-Gaddum-type inequalities for the combination of the Nirmala indices of a graph and its complement were presented with the help of previously deduced relations.

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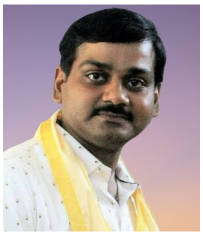

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