TWMS J. App. and Eng. Math. V.15, N.1, 2025, pp. 137-153

CONTINUITY AND BOUNDEDNESS OF LINEAR OPERATORS ON NEUTROSOPHIC 2-NORMED SPACES

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Abstract. In present work, we aim to introduce certain concepts of continuity that is weak, strong and sequentially continuity of linear operators defined on neutrosophic 2-normed spaces. We provide an example that shows sequential continuous linear operators may not be strongly continuous on these spaces. Later, we define weakly and strongly boundedness of an operator on neutrosophic 2-normed spaces and study some relevant connections between continuity and boundedness.

Keywords: Continuity, boundedness, linear operators, sequentially continuity and neutrosophic normed spaces.

AMS Subject Classification: 40A99, 40A05

1. Introduction

In our real life there exist some classes such as the class of beautiful women, the class of intelligent students and the class of tall persons which cannot be fit in the framework of crisp sets. Therefore to explore these type of phenomena's, Zadeh [33] introduced the idea of fuzzy sets by defining the degree of a membership function. Later, the theory of fuzzy sets has grown up along with time and many fuzzy analogues of classical concepts have came into existence. One among these is the fuzzy topology which has wide applications in quantum physics (see [11], [12] and [13]). While studying fuzzy topological space in 1984, Katsaras [19] introduced the concepts of fuzzy semi norm, fuzzy norm and studied some properties of fuzzy semi normed and fuzzy normed spaces. Xiao and Zhu [32] define a fuzzy norm of a linear operator and studied the space of all bounded linear operators endowed with this fuzzy norm. Subsequently, Bag and Samanta [5-6] introduced strong and weak boundedness of fuzzy bounded linear operators and studied their relations with fuzzy continuity. For more information in this direction, we refer to the reader [7] and [8].

Atanassov [2-4] first observed that the Zadeh's idea of fuzzy sets is not sufficient to work on some problems and therefore he generalized it by joining the non-membership

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§ Manuscript received: June 22, 2023; accepted: October 18, 2023.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.1; © Işık University, Department of Mathematics, 2025; all rights reserved.

function along with a membership function. He called it as intutionistic fuzzy set. These sets are further used to define intutionistic topological spaces, intutionistic normed spaces and intutionistic fuzzy 2-normed spaces. For a wide view on these spaces, we quote [14], [18], [17] [9], [10], [22] and [26].

There are situations which are partially true, partially false and partially indeterminacy (neither true nor false) which cannot be modeled by intutionistic fuzzy sets. So, in view of this, Smarandache [30-31] generalized the intutionistic fuzzy sets and define the neutrosophic set by adding the indeterminacy function to the membership and non-membership function. Recently, Kirişci and Şimşek [20] used neutrosophic sets to define neutrosophic normed spaces and studied statistical convergence in these spaces. For some further works on neutrosophic normed spaces, we refer to [27], [16], [21],[23-25], [28] and [29]. In present study, we consider neutrosophic normed spaces and define certain kinds of continuity and boundedness of an operator over neutrosophic 2-normed spaces. We shall also study some interesting relationships among these notions.

2. Preliminaries

This section begins by recalling some definitions and results in concern of present study.

For any set S, the neutrosophic set A of S is defined by $A = \{(s, G_A(s), B_A(s), Y_A(s))\}s \in S\}$, where the functions $G_A : S \to [0, 1]$ and $B_A : S \to [0, 1]$, $Y_A : S \to [0, 1]$ respectively denote the degree of membership function, indeterminacy function and non-membership function of the element of $s \in S$ and for every $s \in S$, $0 \le G(s) + B(s) + Y(s) \le 1$.

Definition 2.1 [1] Let $I = [0, 1]$. A function $\circ : I \times I \to I$ is said to be a $t-$norm for all $f, g, h, i \in I$ we have:

(i) $f \circ g = g \circ f$;

(ii)$f \circ (g \circ h) = (f \circ g) \circ h$;

(iii) $\circ$ is continuous;

(iv) $f \circ 1 = f$ for every $f \in [0,\ 1]$ and

(v) $f \circ g \le h \circ i$ whenever $f \le h$ and $g \le i$.

Definition 2.2 [1] Let $I = [0, 1]$. A function $\diamond : I \times I \to I$ is said to be a continuous triangular conorm or $t-$conorm for all $f, g, h, i \in I$ we have:

(i) $f \diamond g = g \diamond f$;

(ii)$f \diamond (g \diamond h) = (f \diamond g) \circ h$;

(iii) $\diamond$ is continuous;

(iv) $f \diamond 0 = f$ for every $f \in [0, 1]$

(v) $f \diamond g \le h \diamond i$ whenever $f \le h$ and $g \le i$.

We now recall the concept of 2-norm given in [15].

Definition 2.3 Let F be a $d-$dimensional real vector space, where $2 \le d < \infty$. A $2-$norm on F is a function $\|.,.\| : F \times F \to \mathbb{R}$ fulfilling the below listed requirements:

For all $p, q \in F$, and scalar α, we have

(i) $\|p, q\| = 0$ iff p and q are linearly dependent;

(ii) $\|p, q\| = \|p, q\|$;

(iii)$\|\alpha p, q\| = ||\alpha|| \|p, q\|$ and

(iv) $\|p, q + r\| \le \|p, q\| + \|p, r\|$.

The pair $(F, \|.,.\|)$ is known as $2-$normed space in this case.

Let $F = \mathbb{R}^2$ and for $p = (p_1, p_2)$ and $q = (q_1, q_2)$ we define $\|p, q\| = |p_1q_2 - p_2q_1|$, then $\|p, q\|$ is a $2-$ norm on $F = \mathbb{R}^2$.

Kirişci and Şimşek [20] recently defined neutrosophic normed space where as the concept has been extanded for neutrosophic-2-normed linear spaces (briefly abbreviated as $N-2-NS$) in [21] as follows.

Definition 2.4 A six-tuple $V = (F, G, B, Y, \circ, \diamond)$ where F is a vector space, $\circ$ is a $t-$norm, $\diamond$ is a $t-$conorm and G, B, Y are fuzzy sets on $F^2 \times [0,1]$ (G is the membership function, B is the indeterminacy function and Y is the non-membership function) is called a neutrosophic $2-$norm space (briefly $N-2-NS$) if for every $p, q, w \in V$, $\rho, \mu \geq 0$ and $\varsigma \neq 0$ the following conditions are satisfied.

(i) $0 \leq G(p, q; \rho) \leq 1$, $0 \leq B(p, q; \rho) \leq 1$ and $0 \leq Y(p, q; \rho) \leq 1$ for every $\rho \in R^+$;
(ii) $0 \leq G(p, q; \rho) + B(p, q; \rho) + Y(p, q; \rho) \leq 3$;
(iii) $G(p, q; \rho) = 1$ iff p, q are linearly dependent;
(iv) $G(\varsigma p,\ q;\ \rho) = G(p, q; \frac{\rho}{|\varsigma|})$ for each $\varsigma \neq 0$;
(v) $G(p, q; \rho) \circ G(p, w; \mu) \leq G(p, q + w; \rho + \mu)$;
(vi) $G(p, q; .) : [0, \infty) \to [0, 1]$ is a non-decreasing function that runs continuously;
(vii) $\lim_{\rho \to \infty} G(p, q; \rho) = 1$;
(viii) $G(p, q; \rho) = G(q, p; \rho)$
(ix) $B(p, q; \rho) = 0$ iff p, q are linearly dependent;
(x) $B(\varsigma p, q; \rho) = B(p, q; \frac{\rho}{|\varsigma|})$ for each $\varsigma \neq 0$;
(xi) $B(p, q; \rho) \diamond B(p, w; \mu) \geq B(p, q + w; \rho + \mu)$;
(xii) $B(p, q; .) : [0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;
(xiii) $\lim_{\rho \to \infty} B(p, q; \rho) = 0$;
(xiv) $B(p, q; \rho) = B(q, p; \rho)$
(xvi) $Y(p, q; \rho) = 0$ iff p, q are linearly dependent;
(xv)$Y(\varsigma p,\ q;\ \rho) = Y(p,\ q; \frac{\rho}{|\varsigma|})$ for each $\varsigma \neq 0$;
(xvi) $Y(p, q; \rho) \diamond Y(p, w; \mu) \geq Y(p, q + w; \rho + \mu)$;
(xvii) $Y(p, q; .) : [0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;
(xviii) $\lim_{\rho \to \infty} Y(p, q; \rho) = 0$;
(xix) $Y(p, q; \rho) = Y(q, p; \rho)$
(xx) if $\rho \leq 0$, then $G(p, q; \rho) = 0$, $B(p, q; \rho) = 1$, $Y(p, q; \rho) = 1$.

In this case, we call $N_2(G, B, Y)$ a neutrosophic $2-$norm on F.

We next give the notions of convergence in neutrosophic 2-norm space.

Definition 2.5 [21] Let V be a $N-2-NS$. Choose $0 < \epsilon < 1$ and $\rho > 0$. A sequence (v_k) in a V is said to be convergent if $\exists$ a positive integer m and $v_0 \in F$ s.t. $G(v_k - v_0, w; \rho) > 1 - \epsilon$ and $B(v_k - v_0, w; \rho) < \epsilon$, $Y(v_k - v_0, w; \rho) < \epsilon$ for all $k \geq m$ and $w \in V$. This is equivalently to say $\lim_{k\to\infty} G(v_k - v_0, w; \rho) = 1$, $\lim_{k\to\infty} B\left(v_k - v_0, w; \rho\right) = 0$ and $\lim_{k\to\infty} Y\left(v_k - v_0, w; \rho\right) = 0$. In this case, we write $N_2(G, B, Y) - \lim_{k\to\infty} v_k = v_0$.

3. Main Results

Let, $U = (X, G_1, B_1, Y_1, \circ_1, \diamond_1)$ and $V = (Y, G_2, B_2, Y_2, \circ_2, \diamond_2)$ be two neutrosophic 2-normed spaces, where X and Y are linear space over $\mathbb{R}$.

Definition 3.1 A mapping $T : U \to V$ is said to be neutrosophic continuous at $u_0 = \left(u_0^1, u_0^2\right) \in X^2$ if for $\epsilon > 0$ and $\eta > 0$ $(0 < \eta < 1)$, $\exists\ \delta = \delta(\eta, \epsilon) > 0$ and $\xi = \xi(\eta, \epsilon) > 0$ s.t, $\forall\ u = (u_1, u_2) \in X^2$ we have

$$G_1\Bigg((u_1, u_2) - \left(u_1^0, u_2^0\right), \delta\Bigg) > \xi \text{ and } B_1\Bigg((u_1, u_2) - \left(u_1^0, u_2^0\right), \delta\Bigg) < 1 - \xi,$$

$$Y_1\Bigg((u_1, u_2) - (u_1^0, u_2^0), \delta\Bigg) < 1 - \xi,$$

$\Rightarrow$

$$G_2\bigg(T(u_1,u_2)-T\big(u_1^0,u_2^0\big),\epsilon\bigg)>\eta \text{ and } B_2\bigg(T(u_1,u_2)-\big(u_1^0,u_2^0\big),\epsilon\bigg)<1-\eta,$$
$$Y_2\bigg(T(u_1,u_2)-(u_1^0,u_2^0),\epsilon\bigg)<1-\eta.$$

This is equivalent to say that for $\epsilon>0$ and $\eta>0$ $(0<\eta<1)$, $\exists\ \delta=\delta(\eta,\epsilon)>0$ and $\xi=\xi(\eta,\epsilon)>0$ s.t $\forall\ u=(u_1,u_2)\in X^2$.

$$G_1(u-u_0,\delta)>\xi \text{ and } B_1(u-u_0,\delta)<1-\xi,\ Y_1(u-u_0,\delta)<1-\xi,$$
$$\Rightarrow\quad G_2\bigg(T(u)-T(u_0),\epsilon\bigg)>\eta \text{ and } B_2\bigg(T(u)-T(u_0),\epsilon\bigg)<1-\eta,$$
$$Y_2\bigg(T(u)-T(u_0),\epsilon\bigg)<1-\eta.$$

$T:U\to V$ is said to be neutrosophic continuous on U if T is neutrosophic continuous at each point of X^2.

Definition 3.2 A map $T:U\to V$ is said to be strongly neutrosophic continuous at $u_0=\big(u_1^0,u_2^0\big)\in X^2$ if for $\epsilon>0,\ \exists\ \delta>0$ s.t $\forall\ u=(u_1,u_2)\in X^2$

$$G_2\big(T(u)-T(u_0),\epsilon\big)\geq G_1(u-u_0,\delta) \text{ and}$$
$$B_2\big(T(u)-T(u_0),\epsilon\big)\leq B_1(u-u_0,\delta),$$
$$Y_2\big(T(u)-T(u_0),\epsilon\big)\leq Y_1(u-u_0,\delta).$$

$T:U\to V$ is said to be strongly neutrosophic continuous on U if T is strongly neutrosophic continuous at each point of X^2.

Definition 3.3 A map $T:U\to V$ is said to be weakly neutrosophic continuous at $u_0=(u_1^0,u_2^0)\in X^2$ if for $\epsilon>0$ and $\eta\in(0,1),\ \exists\ \delta=(\eta,\epsilon)>0$ s.t $\forall\ u=(u_1,u_2)\in X^2$,

$$G_1(u-u_0,\delta)\geq\eta \text{ and } B_1(u-u_0,\delta)\leq 1-\eta,\ Y_1(u-u_0,\delta)\leq 1-\eta$$
$$\Rightarrow G_2(T(u)-T(u_0),\epsilon)\geq\eta \text{ and } B_2(T(u)-T(u_0),\epsilon)\leq 1-\eta,$$
$$Y_2(T(u)-T(u_0),\epsilon)\leq 1-\eta.$$

We say $T:U\to V$ weakly neutrosophic continuous on U if T is weakly neutrosophic continuous at each point of X^2.

Definition 3.4 A map $T:U\to V$ is said to be sequentially neutrosophic continuous at $u_0=(u_1^0,u_2^0)\in X^2$ if for any sequence (u_k) with $u_k\to u_0$ implies $T(u_k)\to T(u_0)$ i.e, for all $r>0$

$$\lim_{k\to\infty}G_1(u_k-u_0,r)=1 \text{ and } \lim_{k\to\infty}B_1(u_k-u_0,r)=0,\ \lim_{k\to\infty}Y_1(u_k-u_0,r)=0,$$
$$\Rightarrow \lim_{k\to\infty}G_2(T(u_k)-T(u_0),r)=1 \text{ and} \lim_{k\to\infty}B_2(T(u_k)-T(u_0),r)=0,$$
$$\lim_{k\to\infty}Y_2(T(u_k)-T(u_0),r)=0.$$

$T:U\to V$ is said to be sequentially neutrosophic continuous on U if T is sequentially neutrosophic continuous at each point of X^2.

Theorem 3.1 If a map $T:U\to V$ is strongly neutrosophic continuous then it is sequentially neutrosophic continuous.

Proof Let $T : U \to V$ be strongly neutrosophic continuous. We shall show that T is sequentially neutrosophic continuous. Let $u_0 = (u_1^0, u_2^0) \in X^2$ be any point. Since $T : U \to V$ is strongly neutrosophic continuous so for each $\epsilon > 0$, $\exists\ \delta > 0$ s.t $\forall\ u = (u_1, u_2) \in X^2$

$$G_2(T(u) - T(u_0), \epsilon) \geq G_1(u - u_0, \delta) \text{ and } B_2(T(u) - T(u_0), \epsilon) \leq B_1(u - u_0, \delta), \quad Y_2(T(u) - T(u_0), \epsilon) \leq Y_1(u - u_0, \delta) \tag{1}$$

Let (u_k) be any sequence in U s.t $u_k \to u_0$ w.r.t $N_1(G_1, B_1, Y_1)$, then

$$\lim_{k\to\infty} G_1(u_k - u_0, r) = 1 \text{ and } \lim_{k\to\infty} B_1(u_k - u_0, r) = \lim_{k\to\infty} Y_1(u_k - u_0, r) = 0. \tag{2}$$

Now, by (3)

$$G_2(T(u_k) - T(u_0), \epsilon) \geq G_1(u_k - u_0, \delta) \text{ and } B_2(T(u_k) - T(u_0), \epsilon) \leq B_1(u_k - u_0, \delta), \quad Y_2(T(u_k) - T(u_0), \epsilon) \leq Y_1(u_k - u_0, \delta)$$

and therefore,
$\lim_{k\to\infty} G_2(T(u_k) - T(u_0), \epsilon) \geq \lim_{k\to\infty} G_1(u_k - u_0, \delta) = 1$ by (4).
This gives $\lim_{k\to\infty} G_2(T(u_k) - T(u_0), \epsilon) = 1$.
Further,
$\lim_{k\to\infty} B_2(T(u_k) - T(u_0), \epsilon) \leq \lim_{k\to\infty} B_1(u_k - u_0, \delta) = 0$ and
$\lim_{k\to\infty} Y_2(T(u_k) - T(u_0), \epsilon) \leq \lim_{k\to\infty} Y_1(u_k - u_0, \delta) = 0$.
This shows that $T(u_k) \to T(u_0)$ w.r.t $N_2(G_2, B_2, Y_2)$ and therefore T is sequentially neutrosophic continuous. □
The converse of above result is not true in general as can be seen from the following example.
Example 3.1 Let $(X, \|.\|_2)$ be a 2-normed space. Define the $t-$norm, $t-$conorm, $G_1, G_2, B_1, B_2, \&\ Y_1, Y_2$ by
$a \circ b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for $a, b \in [0, 1]$;

$$G_1(u_1, u_2, \delta) = \frac{\delta}{\delta + \|(u_1, u_2)\|_2}, \ B_1(u_1, u_2, \delta) = \frac{\|(u_1, u_2)\|_2}{\delta + \|(u_1, u_2)\|_2}$$
$$Y_1(u_1, u_2, \delta) = \frac{\|(u_1, u_2)\|_2}{\delta};$$
$$G_2(u_1, u_2, \epsilon) = \frac{\epsilon}{\epsilon + \alpha\|(u_1, u_2)\|_2}, \ B_2(u_1, u_2, \epsilon) = \frac{\alpha\|(u_1, u_2)\|_2}{\epsilon + \alpha\|u_1, u_2\|_2}$$
$$Y_2(u_1, u_2, \epsilon) = \frac{\alpha\|(u_1, u_2)\|_2}{\epsilon}$$

where $\epsilon > 0,\ \exists\ \delta > 0,\ \alpha > 0$, and $u = (u_1, u_2) \in X^2$, then $U = (X^2, G_1, B_1, Y_1, \circ, \diamond)$, $V = (X^2, G_2, B_2, Y_2, \circ, \diamond)$ are neutrosophic 2-normed linear spaces. Define a map $T : U \to V$ by $T(u) = \dfrac{u^4}{1 + u^2}$ where $u = (u_1, u_2) \in X^2$. We first show that T is sequentially neutrosophic continuous. Let $u_0 \in U$ and (u_k) be any sequence in U s.t $(u_k) \to u_0$ w.r.t $N_1(G_1, B_1, Y_1)$. Then for any $\delta > 0$, we have

$$\lim_{k\to\infty} G_1(u_k - u_0, \delta) = 1 \text{ and } \lim_{k\to\infty} B_1(u_k - u_0, \delta) = \lim_{k\to\infty} Y_1(u_k - u_0, \delta) = 0.$$
$$\Rightarrow \lim_{k\to\infty} \frac{\delta}{\delta + \|u_k - u_0\|_2} = 1 \text{ and } \lim_{k\to\infty} \frac{\|u_k - u_0\|_2}{\delta + \|u_k - u_0\|_2} = \lim_{k\to\infty} \frac{\|u_k - u_0\|_2}{\delta} = 0,$$

and therefore we have

$$\lim_{k\to\infty} \|u_k - u_0\|_2 = 0. \tag{3}$$

Now consider, $G_2(T(u_k)-T(u_0),\epsilon)=\dfrac{\epsilon}{\epsilon+\alpha\|(T(u_k)-T(u_0)\|_2}=\dfrac{\epsilon}{\epsilon+\alpha\left\|\frac{u_k^4}{1+u_k^2}-\frac{u_0^4}{1+u_0^2}\right\|_2}$

$$=\frac{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2}{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2+\alpha\left\|\left(u_k^4(1+u_0^2)-u_0^4(1+u_k^2)\right)\right\|_2}$$

$$=\frac{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2}{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2+\alpha\|u_k^4+u_k^4u_0^2-u_0^4-u_0^4u_k^2\|_2}$$

$$=\frac{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2}{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2+\alpha\|(u_k-u_0)(u_k+u_0)(u_k^2+u_0^2)+u_k^2u_0^2(u_k^2-u_0^2)\|_2}$$

$$=\frac{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2}{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2+\alpha\|u_k-u_0\|_2\ \|(u_k+u_0)(u_k^2+u_0^2)+u_k^2u_0^2(u_k+u_0)\|_2},$$

and therefore $\lim\limits_{k\to\infty} G_2(T(u_k)-T(u_0),\epsilon)=1.$ by(5)

Further,

$$B_2(T(u_k)-T(u_0),\epsilon)=\frac{\alpha\left\|\frac{u_k^4}{1+u_k^2}-\frac{u_0^4}{1+u_0^2}\right\|_2}{\epsilon+\alpha\left\|\frac{u_k^4}{1+u_k^2}-\frac{u_0^4}{1+u_0^2}\right\|_2}$$

$$=\frac{\alpha\|u_k^4(1+u_0^2)-u_0^4(1+u_k^2)\|_2}{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2+\alpha\|u_k^4(1+u_0^2)-u_0^4(1+u_k^2)\|_2}$$

$$=\frac{\alpha\|u_k^4-u_0^4+u_k^4u_0^2-u_0^4u_k^2\|_2}{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2+\alpha\|u_k^4-u_0^4+u_k^4u_0^2-u_0^4u_k^2\|_2}$$

$$=\frac{\alpha\|(u_k-u_0)(u_k+u_0)(u_k^2+u_0^2)+u_k^2u_0^2(u_k^2-u_0^2)\|_2}{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2+\alpha\|(u_k-u_0)(u_k+u_0)(u_k^2+u_0^2)+u_k^2u_0^2(u_k^2-u_0^2)\|_2}$$

$$=\frac{\alpha\|u_k-u_0\|_2\ \|(u_k+u_0)(u_k^2+u_0^2)+u_k^2u_0^2(u_k+u_0)\|_2}{\epsilon\|1+u_k^2\|_2\ \|1+u_0^2\|_2+\alpha\|u_k-u_0\|_2\ \|(u_k+u_0)(u_k^2+u_0^2)+u_k^2u_0^2(u_k+u_0)\|_2}$$

and therefore $\lim\limits_{k\to\infty} B_2(T(u_K)-T(u_0),\epsilon)=0.$ by(5)

Similarly, we have $\lim\limits_{k\to\infty} Y_2(T(u_K)-T(u_0),\epsilon)=0$, and therefore $T(u_k)\to T(u_0)$ w.r.t $N_2(G_2,B_2,Y_2)$. This shows that T is sequentially neutrosophic continuous on U. We claim that T is not strongly neutrosophic continuous on U. Suppose that T is strongly continuous on U. Let $\epsilon>0$ be given and $u_0=(u_1^0,u_2^0)\in X^2$. Since T is strongly neutrosophic continuous so $\exists\ \delta>0$ s.t $\forall\ u=(u_1,u_2)\in X^2$. $G_2(T(u)-T(u_0),\epsilon)\geq G_1(u-u_0,\delta)$

$$\Rightarrow\ \frac{\epsilon\|1+u^2\|_2\ \|1+u_0^2\|_2}{\epsilon\|1+u^2\|_2\ \|1+u_0^2\|_2+\alpha\|u-u_0\|_2\ \|(u+u_0)(u^2+u_0^2)+u^2u_0^2(u+u_0)\|_2}$$
$$\geq\frac{\delta}{\delta+\|u-u_0\|_2}$$

and
$B_2(T(u)-T(u_0),\epsilon) \leq B_1(u-u_0,\delta)$

$$\Rightarrow \frac{\alpha\|u-u_0\|_2\ \|(u^2+u_0^2)+u^2u_0^2(u+u_0)\|_2}{\epsilon\ \|1+u^2\|_2\ \|1+u_0^2\|_2+\alpha\|u-u_0\|_2\ \|(u+u_0)(u^2+u_0^2)+u^2u_0^2(u+u_0)\|_2} \leq \frac{\|u-u_0\|_2}{\delta+\|u-u_0\|_2}$$

$$\alpha\delta\|u-u_0\|_2\ \|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2+\alpha\|u-u_0\|_2\ \|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2$$
$$\leq \epsilon\|1+u^2\|_2\ \|1+u_0^2\|_2\|u-u_0\|_2+\alpha\|u-u_0\|_2\ \|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2$$
$$\Rightarrow \alpha\delta\|u-u_0\|_2\|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2 \leq \epsilon\|1+u^2\|_2\ \|1+u_0^2\|_2\ \|u-u_0\|_2$$
$$\Rightarrow \delta \leq \frac{\epsilon\|1+u^2\|_2\|1+u_0^2\|_2\ \|u-u_0\|_2}{\alpha\|u-u_0\|_2\ \|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2}$$

$$= \frac{\epsilon\|1+u^2\|_2\|1+u_0^2\|_2}{\alpha\|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2}. \tag{4}$$

$Y_2(T(u)-T(u_0),\epsilon) \leq Y_1(u-u_0,\delta)$

$$\Rightarrow \frac{\alpha\|T(u)-T(u_0)\|_2}{\epsilon} \leq \frac{\|u-u_0\|_2}{\delta}$$
$$\Rightarrow \alpha\left\|\frac{u^4}{1+u^2}-\frac{u_0^4}{1+u_0^2}\right\|_2 \leq \frac{\epsilon}{\delta}\|u-u_0\|_2$$
$$\Rightarrow \frac{\alpha\|u^4(1+u_0^2)-u_0^4(1+u^2)\|_2}{\|1+u^2\|_2\ \|1+u_0^2\|_2} \leq \frac{\epsilon}{\delta}\|u-u_0\|_2$$
$$\Rightarrow \delta\alpha\|u^4-u_0^4+u^4u_0^2-u_0^4u^2\|_2$$
$$\leq \epsilon\|u-u_0\|_2\ \|1+u_o^2\|_2\ \|1+u^2\|_2$$
$$\Rightarrow \delta\alpha\|(u^2-u_0^2)(u^2+u_0^2)+u^2u_0^2(u^2-u_0^2)\|_2$$
$$\leq \epsilon\|u-u_0\|_2\ \|1+u_o^2\|_2\ \|1+u^2\|_2$$
$$\Rightarrow \delta\alpha\|u-u_0\|_2\ \|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2$$
$$\leq \epsilon\|u-u_0\|_2\ \|1+u_o^2\|_2\ \|1+u^2\|_2$$
$$\Rightarrow \delta \leq \frac{\epsilon}{\alpha}\frac{\|1+u_0^2\|_2\ \|1+u^2\|_2}{\|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2}.$$

Hence, in all cases

$$\Rightarrow \delta \leq \frac{\epsilon}{\alpha}\frac{\|1+u_0^2\|_2\ \|1+u^2\|_2}{\|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2}.$$

Let,

$$\delta^* = \inf_{\substack{u \\ u\neq u_0}} \frac{\|1+u_0^2\|_2\ \|1+u^2\|_2}{\|u+u_0\|_2\ \|u^2+u_0^2+u^2u_0^2\|_2},$$

then $\delta = \frac{\epsilon}{\alpha}\delta^*$. But $\delta^* = 0$ which is not possible. Hence, T is not strongly neutrosophic continuous on U. □

Theorem 3.2 A map $T : U \to V$ is neutrosophic continuous if and only if T is sequentially neutrosophic continuous on U.

Proof Suppose $T : U \to V$ is neutrosophic continuous on U. We shall prove that T is sequentially neutrosophic continuous. Let $u_0 \in U$ be any element and $u = (u_k)$ be any

sequence in U converging to u_0 w.r.t $N_1(G_1, B_1, Y_1)$ i.e. $N_1(G_1, B_1, Y_1) - \lim_{k\to\infty} u_k = u_0$.
Let $\epsilon > 0$ and $0 < \eta < 1$.
Since, $T : U \to V$ is neutrosophic continuous at u_0 so $\exists\ \delta = \delta(\eta, \epsilon) > 0$ and $\xi = \xi(\eta, \epsilon) > 0$ s.t for all $u = (u_1, u_2) \in X^2$ satisfying

$$G_1(u - u_0, \delta) > \xi \text{ and } B_1(u - u_0, \delta) < 1 - \xi,\ Y_1(u - u_0, \delta) < 1 - \xi,$$

we have

$$\begin{aligned} &G_2(T(u) - T(u_0), \epsilon) > \eta \text{ and} \\ &\qquad B_2(T(u) - T(u_0), \epsilon) < 1 - \eta,\ Y_2(T(u) - T(u_0), \epsilon) < 1 - \eta. \end{aligned} \tag{5}$$

Since $N_1(G_1, B_1, Y_1) - \lim_{k\to\infty} u_k = u_0$, so $\exists\ k_1 \in \mathbb{N}$ s.t for all $k \geq k_1$, we have

$$G_1(u_k - u_0, \delta) > \xi \text{ and } B_1(u_k - u_0, \delta) < 1 - \xi,\ Y_1(u_k - u_0, \delta) < 1 - \xi.$$

so by (5) we have for all $k \geq k_1$

$$\begin{aligned} &G_2(T(u_k) - T(u_0), \epsilon) > \eta \text{ and} \\ &\qquad B_2(T(u_k) - T(u_0), \epsilon) < 1 - \eta,\ Y_2(T(u_k) - T(u_0), \epsilon) < 1 - \eta. \end{aligned}$$

This show that $T(u_k) \to T(u_0)$ w.r.t $N_2(G_2, B_2, Y_2)$ and therefore T is sequentially neutrosophic continuous on U as u_0 was selected arbitrary.
Conversely, suppose that $T : U \to V$ is sequentially neutrosophic continuous on U. We shall prove that T is neutrosophic continuous on U. Suppose that T is not neutrosophic continuous on U. Then $\exists\ u_0 \in U$ s.t T is not neutrosophic continuous at u_0. Then $\exists\ \epsilon > 0$ and $\eta > 0$ s.t for any $\delta > 0$ & $0 < \xi < 1$ there exists $u' \in X^2$ s.t

$$G_1(u_0 - u', \delta) > \xi \text{ and } B_1(u_0 - u', \delta) < 1 - \xi,\ Y_1(u_0 - u', \delta) < 1 - \xi,$$

we have

$$\begin{aligned} &G_2(T(u_0) - T(u'), \epsilon) \leq \eta \text{ and} \\ &\qquad B_2(T(u_0) - T(u'), \epsilon) \geq 1 - \eta,\ Y_2(T(u_0) - T(u'), \epsilon) \geq 1 - \eta. \end{aligned} \tag{6}$$

If we select $\xi = 1 - \frac{1}{k+1}$ and $\delta = \frac{1}{k+1},\ k = 1, 2, 3, \ldots$, then we have a sequence (u'_k) s.t

$$\begin{aligned} &G_1\left(u_0 - u'_k, \frac{1}{k+1}\right) > 1 - \frac{1}{k+1} \text{ and} \\ &\qquad B_1\left(u_0 - u'_k, \frac{1}{k+1}\right) < \frac{1}{k+1},\ Y_1\left(u_0 - u'_k, \frac{1}{k+1}\right) < \frac{1}{k+1}, \end{aligned} \tag{7}$$

but

$$\begin{aligned} &G_2(T(u_0) - T(u'), \epsilon) \leq \eta \text{ and} \\ &\qquad B_2(T(u_0) - T(u'), \epsilon) \geq 1 - \eta,\ Y_2(T(u_0) - T(u'), \epsilon) \geq 1 - \eta. \end{aligned}$$

Further, for $\delta > 0$, we can choose $k_1 \in \mathbb{N}$ s.t for all $k \geq k_1$ we have $\frac{1}{k+1} < \delta$.
Now,

$$\begin{aligned} G_1(u_0 - u'_k, \delta) &\geq G_1\left(u_0 - u'_k, \frac{1}{k+1}\right) > 1 - \frac{1}{k+1} \text{ and} \\ B_1(u_0 - u'_k, \delta) &\leq B_1\left(u_0 - u'_k, \frac{1}{k+1}\right) < \frac{1}{k+1}, \\ Y_1(u_0 - u'_k, \delta) &\leq Y_1\left(u_0 - u'_k, \frac{1}{k+1}\right) < \frac{1}{k+1}. \quad \text{using (7)} \end{aligned}$$

will imply

$$\lim_{k\to\infty} G_1\ (u_0 - u_k^{'}, \delta) = 1 \text{ and } \lim_{k\to\infty} B_1\ (u_0 - u_k^{'}, \delta) = \lim_{k\to\infty} Y_1\ (u_0 - u_k^{'}, \delta) = 0.$$

Thus show that $(u_k^{'}) \to u_0$ w.r.t $N_1(G_1, B_1, Y_1)$.
Now by (6)

$$G_2(T(u_0) - T(u_k^{'}), \epsilon) \leq \eta \text{ and}$$
$$B_2(T(u_0) - T(u_k^{'}), \epsilon) \geq 1 - \eta,\ Y_2(T(u_0) - T(u_k^{'}), \epsilon) \geq 1 - \eta.$$

$$\Rightarrow \lim_{k\to\infty} G_2(T(u_0) - T(u_k^{'}), \epsilon) \neq 1 \text{ and}$$
$$\lim_{k\to\infty} B_2(T(u_0) - T(u_k^{'}), \epsilon) \neq 0,\ \lim_{k\to\infty} Y_2(T(u_0) - T(u_k^{'}), \epsilon) \neq 0$$

and so $T(u_k^{'}) \nrightarrow T(u_0)$ w.r.t $N_2(G_2, B_2, Y_2)$. This show that T is not sequentially continuous as $(u_k^{'}) \to u_0$ w.r.t $N_1(G_1, B_1, Y_1)$ thus, we obtain a contradiction therefore T is neutrosophic continuous on U. □

4. Neutrosophic bounded linear Operators

In this section, we define neutrosophic weak and strong boundedness of a linear operator and study some relevant connections.

Definition 4.1 A linear operator $T : U \to V$ is said to be strongly neutrosophic bounded on U if and only if $\exists\ M > 0$ s.t for all $u \in U$ and $\eta > 0$

$$G_2(T(u), \eta) \geq G_1\left(u, \frac{\eta}{M}\right) \text{ and } B_2(T(u), \eta) \leq B_1\left(u, \frac{\eta}{M}\right),$$
$$Y_2(T(u), \eta) \leq Y_1\left(u, \frac{\eta}{M}\right).$$

Example 4.1 Let $(X, \|.\|_2)$ be a 2-normed linear space. Define G_1, G_2, B_1, B_2 and Y_1, Y_2 as follows.

$$G_1(u_1, u_2, \eta) = \begin{cases} \frac{\eta}{\eta + \alpha_1 \|u_1, u_2\|_2} & \text{if } \eta > 0 \\ 0 & \text{if } \eta \leq 0\ ; \end{cases}$$

$$B_1(u_1, u_2, \eta) = \begin{cases} \frac{\alpha_1 \|u_1, u_2\|_2}{\eta + \alpha_1 \|u_1, u_2\|_2} & \text{if } \eta > 0 \\ 0 & \text{if } \eta \leq 0\ ; \end{cases} \quad Y_1(u_1, u_2, \eta) = \begin{cases} \frac{\alpha_1 \|u_1, u_2\|_2}{\eta} & \text{if } \eta > 0 \\ 0 & \text{if } \eta \leq 0\ ; \end{cases}$$

and

$$G_2(u_1, u_2, \eta) = \begin{cases} \frac{\eta}{\eta + \alpha_2 \|u_1, u_2\|_2} & \text{if } \eta > 0 \\ 0 & \text{if } \eta \leq 0\ ; \end{cases}$$

$$B_2(u_1, u_2, \eta) = \begin{cases} \frac{\alpha_2 \|u_1, u_2\|_2}{\eta + \alpha_1 \|u_1, u_2\|_2} & \text{if } \eta > 0 \\ 0 & \text{if } \eta \leq 0\ ; \end{cases} \quad Y_2(u_1, u_2, \eta) = \begin{cases} \frac{\alpha_2 \|u_1, u_2\|_2}{\eta} & \text{if } \eta > 0 \\ 0 & \text{if } \eta \leq 0\ ; \end{cases}$$

if $\eta > 0$ and G_1, G_2, B_1, B_2 and Y_1, Y_2 are defined to be zero of $\eta \leq 0$, where α_1 and α_2 are fixed positive real numbers and $\alpha_1 > \alpha_2$. It is clear that $(X, G_1, B_1, Y_1, \circ, \diamond)$ and $(X, G_2, B_2, Y_2, \circ, \diamond)$ become $N - 2NLS$. Define an operator $T : (X, G_1) \to (X, G_2)$ by $T(u) = lu$, where $u = (u_1, u_2) \in X^2$, where $l \neq 0 \in \mathbb{R}$ is fixed, then it is easy to see that T is a linear operator. Choose M s.t $M \geq |l|$. Then we have

$$G_2(T(u_1, u_2), \eta) \geq G_1\left(u_1, u_2, \frac{\eta}{M}\right) \ \forall \ (u_1, u_2) \in X, \forall \ \eta \in \mathbb{R}, \tag{8}$$

Since, $u = (u_1, u_2) \in U,\ M \geq |l|$ we have, $\alpha_1 M \geq \alpha_2 |l|$ since $(\alpha_1 > \alpha_2 > 0)$

$$\begin{aligned}
&\Rightarrow \alpha_1 M \|u_1, u_2\|_2 \geq \alpha_2 |l| \|u_1, u_2\|_2 \\
&\Rightarrow \eta + \alpha_1 M \|u_1, u_2\|_2 \geq \eta + \alpha_2 |l| \|u_1, u_2\|_2 \ \forall \ \eta > 0 \\
&\Rightarrow \frac{1}{\eta + \alpha_2 |l| \|u_1, u_2\|_2} \geq \frac{1}{\eta + \alpha_1 M \|u_1, u_2\|_2} \\
&\Rightarrow \frac{\eta}{\eta + \alpha_2 \|l u_1, u_2\|_2} \geq \frac{\eta}{\eta + \alpha_1 M \|u_1, u_2\|_2} \\
&\Rightarrow \frac{\eta}{\eta + \alpha_2 \|l u_1, u_2\|_2} \geq \frac{\frac{\eta}{M}}{\frac{\eta}{M} + \alpha_1 \|u_1, u_2\|_2} \\
&G_2(T(u_1, u_2), \eta) \geq G_1\left(u_1, u_2, \frac{\eta}{M}\right) \ \forall \ \eta > 0 \text{ and } u = (u_1, u_2) \in X.
\end{aligned}$$

Further,

$$\begin{aligned}
&\alpha_2 |l| \leq \alpha_1 M \ \Rightarrow \alpha_2 |l| \eta \leq \alpha_1 M \eta \\
&\Rightarrow \alpha_2 |l| \eta + \alpha_1 \alpha_2 M |l| \|u_1, u_2\|_2 \leq \alpha_1 M \eta + \alpha_1 \alpha_2 M |l| \|u_1, u_2\|_2 \\
&\Rightarrow \alpha_2 |l| (\eta + \alpha_1 M \|u_1, u_2\|_2) \leq \alpha_1 M (\eta + \alpha_2 |l| \|u_1, u_2\|_2) \\
&\Rightarrow \frac{\alpha_2 |l|}{\eta + \alpha_2 |l| \|u_1, u_2\|_2} \leq \frac{\alpha_1 M}{\eta + \alpha_1 M \|u_1, u_2\|_2} \\
&\Rightarrow \frac{\alpha_2 |l| \|u_1, u_2\|_2}{\eta + \alpha_2 |l| \|u_1, u_2\|_2} \leq \frac{\alpha_1 M \|u_1, u_2\|_2}{\eta + \alpha_1 M \|u_1, u_2\|_2} \\
&\Rightarrow \frac{\alpha_2 \|l u_1, u_2\|_2}{\eta + \alpha_2 \|l u_1, u_2\|_2} \leq \frac{\alpha_1 \|u_1, u_2\|_2}{\frac{\eta}{M} + \alpha_1 \|u_1, u_2\|_2}
\end{aligned}$$

$$B_2(T(u_1, u_2), \eta) \leq B_1\left(u_1, u_2, \frac{\eta}{M}\right) \ \forall \ u = (u_1, u_2) \in X, \forall \ \eta \in \mathbb{R},$$

Similarly,

$$Y_2(T(u_1, u_2), \eta) \leq Y_1\left(u_1, u_2, \frac{\eta}{M}\right) \ \forall \ u = (u_1, u_2) \in X, \forall \ \eta \in \mathbb{R}.$$

This shows that the operator T is strongly neutrosophic bounded.

Definition 4.2 A linear operator $T : U \to V$ is said to be weakly neutrosophic bounded on U if for any $\eta, 0 < \eta < 1,\ \exists\ M_\eta > 0$ s.t $\forall u \in U$ and $\xi > 0$

$$G_1\left(u, \frac{\xi}{M_\eta}\right) \geq \eta \text{ and } B_1\left(u, \frac{\xi}{M_\eta}\right) \leq 1 - \eta, Y_1\left(u, \frac{\xi}{M_\eta}\right) \leq 1 - \eta.$$

$\Rightarrow G_2(T(u), \xi) \geq \eta$ and $B_2(T(u), \xi) \leq 1 - \eta, Y_2(T(u), \xi) \leq 1 - \eta$.

Example 4.2 Let $(X, \|.\|_2)$ be a 2-normed space. Define $a \circ b = \min\{a, b\}$, $a \diamond b =$

$\max\{a, b\}$ for $a, b \in [0, 1]$;

$$G_1(u_1, u_2, \xi) = \begin{cases} \frac{\xi^2 - (\|u_1,u_2\|_2)^2}{\xi^2 + (\|u_1,u_2\|_2)^2} & \text{if } \xi > \|u_1, u_2\| \\ 0 & \text{if } \xi \le \|u_1, u_2\| \end{cases};$$

$$B_1(u_1, u_2, \xi) = \begin{cases} \frac{2(\|u_1,u_2\|_2)^2}{\xi^2 + (\|u_1,u_2\|_2)^2} & \text{if } \xi > \|u_1, u_2\| \\ 0 & \text{if } \xi \le \|u_1, u_2\| \end{cases};$$

$$Y_1(u_1, u_2, \xi) = \begin{cases} \frac{2(\|u_1,u_2\|_2)^2}{\xi^2} & \text{if } \xi > \|u_1, u_2\| \\ 0 & \text{if } \xi \le \|u_1, u_2\| \end{cases}; \text{ and}$$

$$G_2(u_1, u_2, \xi) = \begin{cases} \frac{\xi}{\xi + \|u_1,u_2\|_2} & \text{if } \xi > 0, \forall\ u_1, u_2 \in X \\ 0 & \text{if } \xi \le 0, \forall\ u_1, u_2 \in X \end{cases};$$

$$B_2(u_1, u_2, \xi) = \begin{cases} \frac{\|u_1,u_2\|_2}{\xi + \|u_1,u_2\|_2} & \text{if } \xi > 0, \forall\ u_1, u_2 \in X \\ 0 & \text{if } \xi \le 0, \forall\ u_1, u_2 \in X \end{cases};$$

$$Y_2(u_1, u_2, \xi) = \begin{cases} \frac{\|u_1,u_2\|_2}{\xi} & \text{if } \xi > 0, \forall\ u_1, u_2 \in X \\ 0 & \text{if } \xi \le 0, \forall\ u_1, u_2 \in X \end{cases};$$

If $\xi > 0$ and G_1, G_2, B_1, B_2, Y_1 and Y_2 are said to be zero for $\xi \le 0$. Then it easy to see that $U = (X^2, G_1, B_1, Y_1, \circ, \diamond)$ and $V = (X^2, G_2, B_2, Y_2, \circ, \diamond)$ are $N - 2NLS$.

Define an operator $T : U \to V$ by $T(u) = u$ where $u = (u_1, u_2) \in X^2$. If we choose $M_\eta = \frac{1}{1-\eta}\ \forall\ \eta \in (0, 1)$, then for $\xi > \|u_1, u_2\|_2$ we have

$$G_1\left(u_1, u_2, \frac{\xi}{M_\eta}\right) \ge \eta\ \Rightarrow \frac{\xi^2(1-\eta)^2 - (\|u_1, u_2\|_2)^2}{\xi^2(1-\eta)^2 + (\|u_1, u_2\|_2)^2} \ge \eta$$

$$\begin{aligned} \xi^2(1-\eta)^2 - (\|u_1, u_2\|_2)^2 &\ge \eta\xi^2(1-\eta)^2 + \eta(\|u_1, u_2\|_2)^2 \\ \Rightarrow \xi^2(1-\eta)^2 - \eta\xi^2(1-\eta)^2 &\ge (\|u_1, u_2\|_2)^2 + \eta(\|u_1, u_2\|_2)^2 \\ \Rightarrow \xi^2(1-\eta)^2(1-\eta) &\ge (1+\eta)(\|u_1, u_2\|_2)^2 \end{aligned}$$

$$\Rightarrow \xi^2(1-\eta)^3 \ge (1+\eta)(\|u_1, u_2\|_2)^2\ \ \Rightarrow \frac{\xi^2(1-\eta)^3}{(1+\eta)} \ge (\|u_1, u_2\|_2)^2$$

$$\Rightarrow (\|u_1, u_2\|_2)^2 \le \frac{\xi^2(1-\eta)^3}{(1+\eta)}\ \ \Rightarrow \|u_1, u_2\|_2 \le \frac{\xi(1-\eta)^{\frac{3}{2}}}{(1+\eta)^{\frac{1}{2}}}$$

$$\Rightarrow \|u_1, u_2\|_2 \le \frac{\xi(1-\eta)(1-\eta)^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{2}}}\ \ \Rightarrow \xi + \|u_1, u_2\|_2 \le \frac{\xi(1-\eta)(1-\eta)^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{2}}} + \xi$$

$$\Rightarrow \xi + \|u_1, u_2\|_2 \le \frac{\xi(1-\eta)(1-\eta)^{\frac{1}{2}} + \xi(1+\eta)^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{2}}}$$

$$\Rightarrow \xi + \|u_1, u_2\|_2 \le \frac{\xi[(1-\eta)(1-\eta)^{\frac{1}{2}} + (1+\eta)^{\frac{1}{2}}]}{(1+\eta)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{\xi + \|u_1, u_2\|_2}{\xi} \le \frac{(1-\eta)(1-\eta)^{\frac{1}{2}} + (1+\eta)^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{\xi}{\xi + \|u_1, u_2\|_2} \geq \frac{(1+\eta)^{\frac{1}{2}}}{(1-\eta)(1-\eta)^{\frac{1}{2}} + (1+\eta)^{\frac{1}{2}}} \tag{9}$$

Now,

$$\Rightarrow \frac{(1+\eta)^{\frac{1}{2}}}{(1-\eta)(1-\eta)^{\frac{1}{2}} + (1+\eta)^{\frac{1}{2}}} \geq \eta$$
$$\Rightarrow (1+\eta)^{\frac{1}{2}} \geq \eta(1-\eta)(1-\eta)^{\frac{1}{2}} + \eta(1+\eta)^{\frac{1}{2}}$$
$$\Rightarrow (1+\eta)^{\frac{1}{2}} - \eta(1+\eta)^{\frac{1}{2}} \geq \eta(1-\eta)(1-\eta)^{\frac{1}{2}}$$
$$\Rightarrow (1-\eta)(1+\eta)^{\frac{1}{2}} \geq \eta(1-\eta)(1-\eta)^{\frac{1}{2}}$$
$$\Rightarrow (1+\eta)^{\frac{1}{2}} \geq \eta(1-\eta)(1-\eta)^{\frac{1}{2}} \text{ (squaring both sides)}$$
$$\Rightarrow (1+\eta) \geq \eta^2(1-\eta) \quad \Rightarrow 1+\eta \geq \eta^2 - \eta^3 \quad \Rightarrow 1+\eta+\eta^3 \geq \eta^2.$$

This is true $\forall\ \eta \in (0,1)$ by (9) we get,
$G_2(T(u_1,u_2),\xi) \geq \eta$ if $\xi > \|u_1,u_2\|_2$. Since, $\xi \leq \|u_1,u_2\|_2$, $\frac{\xi^2 - k(\|u_1,u_2\|_2)^2}{\xi^2 + k(\|u_1,u_2\|_2)^2} = 0$, it means that

$$G_1(u_1,u_2,\frac{\xi}{M_\eta}) \geq \eta \Rightarrow G_2(T(u_1,u_2),\xi) \geq \eta\ \forall\ \eta \in (0,1).$$

For all cases, we get,

$$G_1(u_1,u_2,\frac{\xi}{M_\eta}) \geq \eta \Rightarrow G_2(T(u_1,u_2),\xi) \geq \eta\ \forall\ \eta \in (0,1).$$

Now

$$B_1(u_1,u_2,\frac{\xi}{M_\eta}) \leq 1-\eta \Rightarrow B_2(T(u_1,u_2),\xi) \leq 1-\eta\ \forall\ \eta \in (0,1);$$

$B_1(u_1,u_2,\frac{\xi}{M_\eta}) \leq 1-\alpha \Rightarrow \frac{2\|x\|^2}{\xi^2(1-\alpha)^2 + \|x\|^2} \leq 1-\alpha$
$2\|x\|^2 \leq (1-\alpha)\bigg(\xi^2(1-\alpha)^2 + \|x\|^2\bigg) \Rightarrow 2\|x\|^2 \leq (1-\alpha)(\xi^2(1-\alpha)^2 + (1-\alpha)\|x\|^2)$
$\Rightarrow 2\|x\|^2 - (1-\alpha)\|x\|^2 \leq (1-\alpha)^3\xi^2 \Rightarrow 2\|x\|^2 - \|x\|^2 + \alpha\|x\|^2 \leq (1-\alpha)^3\xi^2$
$\Rightarrow \|x\|^2 + \alpha\|x\|^2 \leq (1-\alpha)^3\xi^2 \Rightarrow (1+\alpha)\|x\|^2 \leq (1-\alpha)^3\xi^2 \Rightarrow \|x\|^2 \leq \frac{(1-\alpha)^3\xi^2}{(1+\alpha)}$
$\Rightarrow \|x\| \leq \frac{(1-\alpha)^{\frac{3}{2}}\xi}{(1+\alpha)^{\frac{1}{2}}} \Rightarrow \|x\| \leq \frac{(1-\alpha)(1-\alpha)^{\frac{1}{2}}\xi}{(1+\alpha)^{\frac{1}{2}}} \Rightarrow \xi + \|x\| \leq \frac{(1-\alpha)(1-\alpha)^{\frac{1}{2}}\xi}{(1+\alpha)^{\frac{1}{2}}} + \xi$
$\Rightarrow \xi + \|x\| \leq \frac{(1-\alpha)(1-\alpha)^{\frac{1}{2}}\xi + \xi(1+\alpha)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}} \Rightarrow \xi + \|x\| \leq \frac{\xi[(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}]}{(1+\alpha)^{\frac{1}{2}}}$
$\Rightarrow \xi + \|x\| \leq \frac{\|x\|[(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}]}{(1+\alpha)^{\frac{1}{2}}} \Rightarrow \frac{\xi + \|x\|}{\|x\|} \leq \frac{(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}}$
$\Rightarrow \frac{\|x\|}{\xi + \|x\|} \geq \frac{(1+\alpha)^{\frac{1}{2}}}{(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}}$
Now, $\frac{(1+\alpha)^{\frac{1}{2}}}{(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}} \leq (1-\alpha) \Rightarrow (1+\alpha)^{\frac{1}{2}} \leq (1-\alpha)(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1-\alpha)(1+\alpha)^{\frac{1}{2}}$
$\Rightarrow (1+\alpha)^{\frac{1}{2}} - (1-\alpha)(1+\alpha)^{\frac{1}{2}} \leq (1-\alpha)^2(1-\alpha)^{\frac{1}{2}}$
$\Rightarrow (1+\alpha)^{\frac{1}{2}} - (1+\alpha)^{\frac{1}{2}} + \alpha(1+\alpha)^{\frac{1}{2}} \leq (1-\alpha)^2(1-\alpha)^{\frac{1}{2}}$
$\Rightarrow \alpha(1+\alpha)^{\frac{1}{2}} \leq (1-\alpha)^2(1-\alpha)^{\frac{1}{2}}$ (squaring both sides)
$\alpha^2(1+\alpha) \leq (1-\alpha)^4(1-\alpha) \Rightarrow \alpha^2 + \alpha^3 \leq (1-\alpha)^4 - \alpha(1-\alpha)^4$
$\Rightarrow \alpha^2 + \alpha^3 \leq [(1-\alpha)^2]^2 - \alpha[(1-\alpha)^2]^2 \Rightarrow \alpha^2 + \alpha^3 \leq [1-2\alpha+\alpha^2]^2 - \alpha[1-2\alpha+\alpha^2]^2$

$\Rightarrow \alpha^2 + \alpha^3 \leq [1 + 4\alpha^2 + \alpha^4 - 4\alpha - 4\alpha^3 + 2\alpha^2] - \alpha[1 + 4\alpha^2 + \alpha^4 - 4\alpha - 4\alpha^3 + 2\alpha^2]$
$\Rightarrow \alpha^2 + \alpha^3 \leq [1 + 4\alpha^2 + \alpha^4 - 4\alpha - 4\alpha^3 + 2\alpha^2] - \alpha - 4\alpha^3 - \alpha^5 + 4\alpha^2 + 4\alpha^4 - 2\alpha^3$
$\Rightarrow \alpha^2 + \alpha^3 \leq 1 + 10\alpha^2 + 5\alpha^4 - 5\alpha - 10\alpha^3 - \alpha^5$
$\Rightarrow \alpha^3 + 10\alpha^3 + 5\alpha + \alpha^5 \leq 1 + 10\alpha^2 + 5\alpha^4 - \alpha^2$. This is true $\forall\ \eta \in (0, 1)$ we get,
$B_2(T(u_1, u_2), \xi) \leq 1 - \eta$ if $\xi < \|u_1, u_2\|_2$. Since, $\xi \geq \|u_1, u_2\|_2$, $\frac{\xi^2}{\xi^2 + (\|u_1, u_2\|_2)^2} = 0$, it means that

$$B_1\left(u_1, u_2, \frac{\xi}{M_\eta}\right) \leq 1 - \eta \Rightarrow B_2(T(u_1, u_2), \xi) \leq 1 - \eta\ \forall\ \eta \in (0, 1);$$

Similarly,

$$Y_1\left(u_1, u_2, \frac{\xi}{M_\eta}\right) \leq 1 - \eta \Rightarrow Y_2(T(u_1, u_2), \xi) \leq 1 - \eta\ \forall\ \eta \in (0, 1).$$

This shows that T is weakly neutrosophic bounded.

Theorem 4.1 If a linear operator $T : U \to V$ is strongly neutrosophic bounded on U, then it is weakly neutrosophic bounded on U.
Proof Suppose that $T : U \to V$ is strongly neutrosophic bounded on U. So, $\exists\ M > 0$ s.t for all $u \in U$ and $\eta > 0$

$$\begin{gathered} G_2(T(u), \eta) \geq G_1\left(u, \frac{\eta}{M}\right) \text{ and} \\ B_2(T(u), \eta) \leq B_1\left(u, \frac{\eta}{M}\right), Y_2(T(u), \eta) \leq Y_1\left(u, \frac{\eta}{M}\right) \end{gathered} \tag{10}$$

Let, $0 < \xi < 1$, then $\exists\ M_\xi (= M > 0)$ s.t

$$G_1\left(u, \frac{\eta}{M_\xi}\right) \geq \xi \text{ and } B_1\left(u, \frac{\eta}{M_\xi}\right) \leq 1 - \xi, Y_1\left(u, \frac{\eta}{M_\xi}\right) \leq 1 - \xi$$
$$\Rightarrow G_2\big(T(u), \eta\big) \geq G_1\left(u, \frac{\eta}{M_\xi}\right) \geq \xi \text{ and } B_2\big(T(u), \eta\big) \leq B_1\left(u, \frac{\eta}{M_\xi}\right) \leq 1 - \xi,$$
$$Y_2\big(T(u), \eta\big) \leq Y_1\left(u, \frac{\eta}{M_\xi}\right) \leq 1 - \xi. \quad \text{(using (10))}$$

As this holds for all $u \in U$ and $\eta > 0$, therefore $T : U \to V$ is weakly neutrosophic bounded. □

Theorem 4.2 A linear operator $T : U \to V$ is strongly neutrosophic continuous everywhere on U if T is strongly neutrosophic continuous at a point $u_0 \in U$.
Proof Let $u_0 \in U$ be a point in U s.t $T : U \to V$ is strongly neutrosophic continuous at u_0. We shall prove that T is strongly neutrosophic continuous everywhere in U. Since T is strongly neutrosophic continuous at u_0 so for each $\epsilon > 0$, $\exists\ \delta > 0$ s.t

$$\begin{gathered} G_2(T(u) - T(u_0), \epsilon) \geq G_1(u - u_0, \delta) \text{ and } B_2(T(u) - T(u_0), \epsilon) \leq B_1(u - u_0, \delta), \\ Y_2(T(u) - T(u_0), \epsilon) \leq Y_1(u - u_0, \delta). \end{gathered} \tag{11}$$

Let $v \in U$ be any element of U, then $u + u_0 - v$ is also an element of U, and therefore by replacing u by $u+u_0-v$ in (11), we have $G_2(T(u+u_0-v)-T(u_0), \epsilon) \geq G_1(u+u_0-v-u_0, \delta)$ $\Rightarrow G_2(T(u + u_0 - v) - T(u_0), \epsilon) \geq G_1(u - v, \delta)$ i.e., $G_2(T(u) - T(v), \epsilon) \geq G_1(u - v, \delta)$ and $B_2(T(u + u_0 - v) - T(u_0), \epsilon) \leq B_1(u + u_0 - v - u_0, \delta) \Rightarrow B_2(T(u + u_0 - v) - T(u_0), \epsilon) \leq B_1(u-v, \delta)$ i.e., $B_2(T(u)-T(v), \epsilon) \leq B_1(u-v, \delta)$ Similarly, $Y_2(T(u)-T(v), \epsilon) \leq Y_1(u-v, \delta)$.

Since, $v \in U$ was arbitrarily selected so $T : U \to V$ is strongly neutrosophic continuous. $\square$

Theorem 4.3 A linear map $T : U \to V$ is strongly neutrosophic continuous if and only if T is strongly neutrosophic bounded.
Proof Suppose that $T : U \to V$ is strongly neutrosophic continuous on U, then T is strongly neutrosophic continuous at $\theta \in U$ where θ denote the zero element of U. So for $\epsilon = 1,\ \exists\ \delta > 0$ s.t for all $u \in U$

$$G_2(T(u) - T(\theta), 1) \geq G_1(u - \theta, \delta) \text{ and}$$
$$B_2(T(u) - T(\theta), 1) \leq B_1(u - \theta, \delta),\ Y_2(T(u) - T(\theta), 1) \leq Y_1(u - \theta, \delta).$$

Case 1. Let $u \neq \theta$ and $\eta > 0$. Take $v = \frac{u}{\eta}$

$$\begin{aligned} G_2(T(u), \eta) &= G_2(T(\eta v), \eta) = G_2(\eta T(v), \eta) = G_2(T(v), 1) \\ &\geq G_1(v, \delta) = G_1\left(\frac{u}{\eta}, \delta\right) = G_1\left(u, \frac{\eta}{\frac{1}{\delta}}\right) = G_1\left(u, \frac{\eta}{M}\right) \end{aligned}$$

where $M = \frac{1}{\delta}$ i.e $G_2(T(u), \eta) \geq G_1\left(u, \frac{\eta}{M}\right)$ and

$$\begin{aligned} B_2(T(u), \eta) &= B_2(T(\eta v), \eta) = B_2(\eta T(v), \eta) = B_2(T(v), 1) \\ &\leq B_1(v, \delta) = B_1\left(\frac{u}{\eta}, \delta\right) = B_1\left(u, \frac{\eta}{\frac{1}{\delta}}\right) = B_1\left(u, \frac{\eta}{M}\right) \end{aligned}$$

where $M = \frac{1}{\delta}$ i.e $B_2(T(u), \eta) \leq B_1\left(u, \frac{\eta}{M}\right)$; similarly, $Y_2(T(u), \eta) \leq Y_1\left(u, \frac{\eta}{M}\right)$.
Case 2. If $u = \theta$ and $\eta > 0$, then $T(\theta) = \theta$ and

$$G_2(\theta, \eta) = G_1\left(\theta, \frac{\eta}{M}\right) = 1 \text{ and } B_2(\theta, \eta) = B_1\left(\theta, \frac{\eta}{M}\right) = 0,\ Y_2(\theta, \eta) = Y_1\left(\theta, \frac{\eta}{M}\right) = 0.$$

Therefore, in both cases, we have T is strongly neutrosophic bounded.
Conversely, suppose that T is strongly neutrosophic bounded so $\exists\ M > 0$ s.t $\forall\ u \in U$ and $\eta > 0$

$$G_2(T(u), \eta) \geq G_1\left(u, \frac{\eta}{M}\right) \text{ and } B_2(T(u), \eta) \leq B_1\left(u, \frac{\eta}{M}\right),\ Y_2(T(u), \eta) \leq Y_1\left(u, \frac{\eta}{M}\right)$$

Let $\epsilon > 0$, then we have

$$G_2(T(u), \epsilon) \geq G_1\left(u, \frac{\epsilon}{M}\right) \text{ and } B_2(T(u), \epsilon) \leq B_1\left(u, \frac{\epsilon}{M}\right),\ Y_2(T(u), \epsilon) \leq Y_1\left(u, \frac{\epsilon}{M}\right).$$

Take $\delta = \frac{\epsilon}{M}$, then

$$G_2(T(u) - T(\theta), \epsilon) \geq G_1(u - \theta, \delta) \text{ and}$$
$$B_2(T(u) - T(\theta), \epsilon) \leq B_1(u - \theta, \delta),\ Y_2(T(u) - T(\theta), \epsilon) \leq Y_1(u - \theta, \delta),$$

and therefore T is strongly neutrosophic continuous on U. $\square$

Theorem 4.4 If a linear operator $T : U \to V$ is sequentially neutrosophic continuous at u_0 in U then it is sequentially neutrosophic continuous on U.
Proof.
Proof Suppose that $T : U \to V$ is sequentially neutrosophic continuous at u_0 in U. We

shall show that T is sequentially neutrosophic continuous on U. Let $u \in U$ be any arbitrary and (u_k) be any sequence converging to u w.r.t $N_1(G_1, B_1, Y_1)$ then, we have for all $\eta > 0$

$$\lim_{k\to\infty} G_1(u_k - u, \eta) = 1 \text{ and} \lim_{k\to\infty} B_1(u_k - u, \eta) = \lim_{k\to\infty} Y_1(u_k - u, \eta) = 0.$$

This implies that

$$\lim_{k\to\infty} G_1((u_k - u + u_0) - u_0, \eta) = 1 \text{ and}$$
$$\lim_{k\to\infty} B_1((u_k - u + u_0) - u_0, \eta) = \lim_{k\to\infty} Y_1((u_k - u + u_0) - u_0, \eta) = 0.$$

Since T is sequentially neutrosophic continuous at u_0.

$$\lim_{k\to\infty} G_2(T(u_k - u + u_0) - T(u_0), \eta) = 1 \text{ and}$$
$$\lim_{k\to\infty} B_2(T(u_k - u + u_0) - T(u_0), \eta) = \lim_{k\to\infty} Y_2(T(u_k - u + u_0) - T(u_0), \eta) = 0.$$

This gives for each $\eta > 0$

$$\lim_{k\to\infty} G_2(T(u_k) - T(u), \eta) = 1 \text{ and}$$
$$\lim_{k\to\infty} B_2(T(u_k) - T(u), \eta) = 0, \lim_{k\to\infty} Y_2(T(u_k) - T(u), \eta) = 0.$$

This shows that $(T(u_k)) \to T(u)$ w.r.t $N_2(G_2, B_2, Y_2)$ and therefore T is sequentially neutrosophic continuous on U. □

The proof of the following two Theorems is omitted as it can be obtained analogously to the proofs of Theorem 4.2 & Theorem 4.3

Theorem 4.5 A linear operator $T : U \to V$ is weakly neutrosophic continuous on U if T is weakly neutrosophic continuous at a point u_0 in U.
Proof. Omitted.

Theorem 4.6 A linear operator $T : U \to V$ is weakly neutrosophic continuous if and only if T is weakly neutrosophic bounded.
Proof Omitted. (follow the proof Theorem 4.3).

5. Conclusion

Neutrosophic norm is an important generalization of fuzzy norm defined for those problems of real world which seems difficult to solve by crisp norm due to complex indeterminacy and vagueness. In present work we developed some topological aspects of continuity and boundedness in a more general context i.e. in neutrosophic 2-normed space. The results present here will be helpful to develop these spaces mathematically.

Open Problems: Extension of some topological concepts in neutrosophic -n- normed linear spaces.

Acknowledgement

The authors would like to record their gratitude to the reviewers for their careful reading and suggestions which improved the presentation of the paper.

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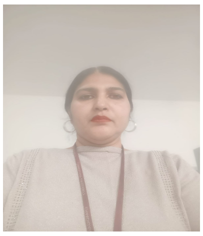

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