

## CONSTRUCTION OF OPTIMAL REGULATORS FOR MULTIPOINT PROBLEMS OF DYNAMIC SYSTEMS

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ABSTRACT. All papers must include an abstract of no more than 120 and no less than 100 words, including articles and prepositions. Authors should avoid complicated mathematical symbols and tables in this part of the paper since it may lead to confusion about the content of the paper when indexing organizations prepare short abstracts.

Keywords: Unmanned aerial vehicle, quadcopter, LQR theory, Riccati equation, motion stabilization.

AMS Subject Classification: 34A30, 49N10, 93C95, 93D20.

### 1. INTRODUCTION

As is known [1-2], when the movement of an object on an infinite time interval  $(0, \infty)$  is described by a system of linear ordinary differential equations

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad (1)$$

and it is required to find the regulation law

$$u = Kx, \quad (2)$$

such that problem (11)-(12) was asymptotically stable and the following quadratic functional [16]

$$J = \frac{1}{2} \int_0^{\infty} (x'Qx + u'Ru)dt, \quad (3)$$

reached minimum value, such a problem is called optimal stabilization by state. Here  $x$  is a  $n$ -dimensional phase vector,  $u$  is the  $m$ -dimensional vector of control influence,  $A$  and  $B$  are the  $n \times n$  and  $n \times m$ -dimensional matrices, respectively, and are a stabilizing pair,  $Q \geq 0$ ,  $R \geq 0$  are symmetric matrices of  $n \times n$  and  $m \times m$  dimension, respectively.

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§ Manuscript received: January 01, 2022; accepted: March 15, 2023.

TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.1; © Işık University, Department of Mathematics, 2025; all rights reserved.

The solution to problem (11)-(13), i.e. the matrix of the feedback circuit  $K$  has the form [3-6]:

$$K = -R^{-1}B^1S, \quad (4)$$

where  $S = S' \geq 0$  is a solution of the algebraic Riccati equation (ARE)

$$SF + F'S - SBR^{-1}B'S + Q = 0. \quad (5)$$

As noted in [7, 8], at a certain time interval, equation (15) has the form

$$\frac{dS}{dt} = SF + F'S - SBR^{-1}BS + Q, \quad (6)$$

where this interval actually describes the transition process, and at point  $\tau$  is fulfilled [7]

$$\left. \frac{dS}{dt} \right|_{t=\tau} \approx 0, \quad (7)$$

and at  $T > \tau$  the system is already in steady state.

Now consider the following optimal control problem on the interval  $(0, T)$ , where the movement of the object is described by equation (11) with boundary conditions

$$x(0) = x_0, \quad x(T) = x_T. \quad (8)$$

It is required to find the control law (12) in such a way that the closed system (11)-(13) satisfies condition (15). Indeed, for a sufficiently large  $T$ , equation (11) has the form

$$(x - x_T)' = A(x - x_T) + Gu, \quad (9)$$

and feedback (12) ensures convergence  $\lim_{t \rightarrow T} x(t) = x_T$  with a certain sufficient accuracy. Note that equation (19) is equivalent to equation

$$\dot{x} = Ax + Gu - Ax_T. \quad (10)$$

Thus, using a given technique, we determine  $T$ , at which  $x(T)$  is approximately equal to  $x_T$ . This technique is applicable to solving multipoint problems of optimization, which is what this article is devoted to. It is assumed that the solution of equation (11) must pass through given points  $x_0, x_1, \dots, x_l$ . To solve such problem, first, for points on the interval  $x_0, x_1$ , the (11)-(13), (18) are solved and the time  $T_1$ , is determined, satisfying condition (17). Thus, taking  $T_0 = 0$  we find a solution on the time interval  $(T_0, T_1)$ , the corresponding trajectory of which connects the points  $x_0$  and  $x_1$ . Further, this technique is extended to points  $x_{i-1}$  and  $x_i$  on time intervals  $(T_{i-1}, T_i)$  for all  $i = 2, \dots, l$ .

## 2. PROBLEM STATEMENT

As is known [8-10, 17-19], the movement of a controlled quadcopter (Fig. 1) is described by the following system of second-order ordinary differential equations:

$$m \ddot{x} = -u \sin\theta,$$

$$m \ddot{y} = u \cos\theta \sin\varphi,$$

$$m \ddot{z} = u \cos\theta \cos\varphi - mg, \quad (11)$$

$$\ddot{\psi} = \tilde{\tau}_\psi,$$

$$\ddot{\theta} = \tilde{\tau}_\theta,$$

$$\ddot{\varphi} = \tilde{\tau}_\varphi.$$



Figure 1. Quadcopter.

Here  $m$  is the quadcopter mass, and control influences are defined as

$$u = f_1 + f_2 + f_3 + f_4,$$

$$\tilde{\tau}_\psi = [(f_2 + f_4) - (f_1 + f_3)]l, \quad (12)$$

$$\tilde{\tau}_\theta = (f_2 - f_4)l,$$

$$\tilde{\tau}_\varphi = (f_3 - f_1)l,$$

where  $l$  is the distance from the center of the quadcopter to the engines, and  $f_i$  is the lifting force of the  $i$ -th engine

$$f_i = k_i \omega_i^2 \quad (13)$$

Here  $\omega_i$  is the angular speed of rotation of the  $i$ -th engine (propeller),  $k_i$  is an experimentally determined constant. Note that the angular speed of rotation of the motor depends on the electric current  $I_i$  supplied to the motor

$$\omega_i = W(I_i) \quad (14)$$

If we divide each side in the first equations of system (11) by  $m$  we get

$$\ddot{x} = -\frac{u}{m} \sin\theta,$$

$$\ddot{y} = \frac{u}{m} \cos\theta \sin\varphi, \quad (15)$$

$$\ddot{z} = \frac{u}{m} \cos\theta \cos\varphi - g.$$

Note that here  $(x, y, z)$  – are the coordinates of the quadcopter's center of gravity, and  $(\psi, \theta, \varphi)$  are Euler angles, i.e. yaw, pitch and roll angles, respectively.

Let's denote

$$\begin{aligned} x(t) = x_1(t), \quad \dot{x}(t) = x_2(t), \quad y(t) = y_1(t), \quad \dot{y}(t) = y_2(t), \\ \dot{z}(t) = z_1(t), \quad \ddot{z}(t) = z_2(t), \quad \theta(t) = \theta_1(t), \quad \dot{\theta}(t) = \theta_2(t), \end{aligned} \quad (16)$$

$$\varphi(t) = \varphi_1(t), \quad \dot{\varphi}(t) = \varphi_2(t), \quad \psi(t) = \psi_1(t), \quad \dot{\psi}(t) = \psi_2(t).$$

Then, taking into account the initial conditions, we obtain the following Cauchy problem

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \quad x_1(0) = x_1^0, \\ \dot{x}_2(t) &= -\frac{u(t)}{m} \sin\theta, \quad x_2(0) = x_2^0, \\ \dot{y}_1(t) &= y_2(t) \quad y_1(0) = y_1^0, \\ \dot{y}_2(t) &= \frac{u(t)}{m} \cos\theta \sin\varphi_1(t) \quad y_2(0) = y_2^0, \\ \dot{z}_1(t) &= z_2(t) \quad z_1(0) = z_1^0, \\ \dot{z}_2(t) &= \frac{u(t)}{m} \cos\theta \cos\varphi_1(t) - gz_2(0) = z_2^0, \\ \dot{\theta}_1(t) &= \theta_2(t) \quad \theta_1(0) = \theta_1^0, \\ \dot{\theta}_2(t) &= \tilde{\tau}_\theta(t) \quad \theta_2(0) = \theta_2^0, \\ \dot{\varphi}_1(t) &= \varphi_2(t) \quad \varphi_1(0) = \varphi_1^0, \\ \dot{\varphi}_2(t) &= \tilde{\tau}_\varphi(t) \quad \varphi_2(0) = \varphi_2^0, \\ \dot{\psi}_1(t) &= \psi_2(t) \quad \psi_1(0) = \psi_1^0, \\ \dot{\psi}_2(t) &= \tilde{\tau}_\psi(t) \quad \psi_2(0) = \psi_2^0. \end{aligned} \tag{17}$$

Let's assume that  $\cos\theta \cdot \cos\varphi \neq 0$ . Next, assuming that the angles  $\theta, \varphi$  are sufficiently small, taking into account (15) we obtain from system (11)

$$\begin{aligned} \ddot{x} &= -g\theta, \\ \ddot{\theta} &= \tilde{\tau}_\varphi, \\ \ddot{y} &= g\varphi, \\ \ddot{\varphi} &= \tilde{\tau}_\varphi, \\ \ddot{z} &= \frac{\tilde{u}}{m} - g = u_z, \\ \ddot{\psi} &= \tilde{\tau}_\psi. \end{aligned} \tag{18}$$

Here  $\tilde{u} = z_1 + mg = u \cos\theta \cos\varphi$ .

In this case, the system of equations (17) can be written in the following form:

$$\dot{x}_1 = x_2,$$



$$P = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \\ \theta_1 \\ \theta_2 \\ \varphi_1 \\ \varphi_2 \\ \psi_1 \\ \psi_2 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\tilde{u}}{m} - g \\ 0 \\ \tilde{\tau}_\theta \\ 0 \\ \tilde{\tau}_\varphi \\ 0 \\ \tilde{\tau}_\psi \end{bmatrix}, \quad P^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ y_1^0 \\ y_2^0 \\ z_1^0 \\ z_2^0 \\ \theta_1^0 \\ \theta_2^0 \\ \varphi_1^0 \\ \varphi_2^0 \\ \psi_1^0 \\ \psi_2^0 \end{bmatrix}, \quad (20)$$

then system (19) can be written in the following compact form:

$$\dot{P} = AP + q, \quad P(0) = P^0. \quad (21)$$

### 3. CONSTRUCTION OF THE REGULATOR

Now let's consider the following problem. We will look for such a vector  $q$ , more precisely, such functions  $\tilde{u}, \tilde{\tau}_\theta, \tilde{\tau}_\varphi, \tilde{\tau}_\psi$ , so that the equality  $P(T) = P^T$  holds. To begin with, let's assume that  $P^T = 0$ . In equation (21), let's the vector  $q$  represent as

$$q = Bu, \quad (22)$$

where

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} u_z \\ \tilde{\tau}_\theta \\ \tilde{\tau}_\varphi \\ \tilde{\tau}_\psi \end{bmatrix}. \quad (23)$$

Then (21) takes the following form [20]

$$\dot{P} = AP + Bu, \quad P(0) = P^0. \quad (24)$$

To synthesize a linear optimal controller of problem (24), we use the following functional as a quality criterion:

$$J = \int_0^\infty (P'QP + u'Ru) dt, \quad (25)$$



Then the initial condition passes to  $\bar{P}(0) = P(0) - P^T = P^0 - P^T = \bar{P}^0$ . Then, similarly to (28), we obtain the following Cauchy problem in the form of a closed system

$$\dot{\bar{P}}(t) = (A + BK)\bar{P}(t), \quad \bar{P}(0) = \bar{P}^0, \tag{33}$$

for which, due to asymptotically stability, we have  $\bar{P}(t) \xrightarrow{t \rightarrow \infty} 0$ , and the control influence has the form  $u = K\bar{P}(t) = K(P(t) - P^T)$ . And this means that  $P(t) \xrightarrow{t \rightarrow \infty} P^T$ . From (33) we easily obtain

$$\dot{P}(t) = (A + BK)(P(t) - P^T), \quad P(0) = P^0. \tag{34}$$

In other words, if equation (33) is replaced by equation

$$\dot{P}(t) = (A + BK)P(t) - (A + BK)P^T, \tag{35}$$

then its solution will satisfy the condition  $P(t) \rightarrow P^T$ . Thus, solving equation (35) with the corresponding initial condition we obtain a solution that satisfies the condition

$$P(t) \xrightarrow{t \rightarrow \infty} P^T, \tag{36}$$

and the control action is determined by the formula

$$u(t) = K(P(t) - P^T). \tag{37}$$

From (36) it follows that for a sufficiently small  $\varepsilon > 0$  one can find such  $T$ , for which  $|P(T) - P^T| < \varepsilon$ . Thus, we can state  $P(T) \approx P^T$  with sufficient accuracy. Note that  $T$  can be found from conditions (17). In this case, it can be argued that the solution to equation (35) will satisfy the boundary condition (18)

$$P(T) \approx P^T.$$

#### 4. REGULATOR FOR MULTI-POINT PROBLEM

Now let's consider the problem of constructing a regulator for controlling the motion of a quadcopter [13-15], the trajectory of which should pass through the points  $P^0, P^1, \dots, P^l$ . Let's first consider the problem

$$\dot{P}_1(t) = (A + BK)(P_1(t) - P^1), \quad P_1(T_0) = P^0. \tag{38}$$

$$J_1 = \int_{T_0}^{\infty} (P_1'QP_1 + u_1'Ru_1) dt, \tag{39}$$

where  $A, B, Q, R$  and  $K$  are determined by the formulas (20), (23), (26) and (31). Solving this synthesis problem as in the previous paragraph and using (17) we can determine a  $T_1 > T_0$ , for which  $P_1(T_1) \approx P^1$ . Thus, we have shown that the solution of problem (38)-(39) at  $u_1(t) = K(P_1(t) - P^1)$  satisfies the boundary conditions  $P_1(T_0) = P^0$  and  $P_1(T_1) = P^1$ . Next, we move on to pairs of points  $(P^1, P^2), \dots, (P^{i-1}, P^i), \dots, (P^{l-1}, P^l)$ . Then for  $i = 2, 3, \dots, l$  we consider the problem

$$\dot{P}_i(t) = (A + BK)(P_i(t) - P^i), \quad P_i(T_{i-1}) = P^{i-1}, \tag{40}$$



$$J_i = \int_{T_{i-1}}^{\infty} (P_i'QP_i + u_i'Ru_i) dt. \tag{41}$$

Solving the synthesis problem (40), (41) for each  $i$  we can determine such  $T_i > T_{i-1}$ , for which  $P_i(T_i) \approx P^i$ . The control action on the interval  $(T_{i-1}, T_i)$  is defined as  $u_i(t) = K(P_i(t) - P^i)$ .

### 5. EXAMPLE

Let  $l = 4$  and points  $P^0, P^1, P^2, P^3, P^4$  are given as follows:

$$P^0 = \begin{bmatrix} 20 \\ 0 \\ -10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad P^1 = \begin{bmatrix} 20 \\ 0 \\ -10 \\ 0 \\ 200 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad P^2 = \begin{bmatrix} 150 \\ 0 \\ -10 \\ 0 \\ 200 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad P^3 = \begin{bmatrix} 150 \\ 0 \\ 100 \\ 0 \\ 200 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad P^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Using the proposed methodology, we have proposed a regulator synthesis algorithm, with the help of which the trajectory of the center of gravity of the controlled quadcopter is found (Fig. 2).

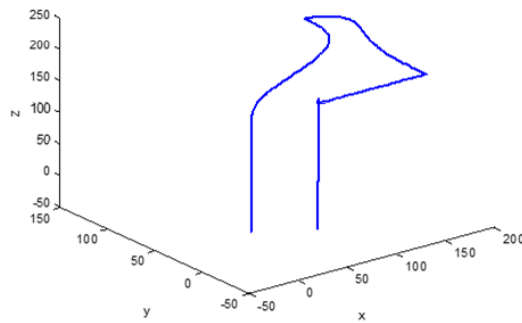


Figure 1. Trajectory of the center of gravity of a controlled quadcopter.

Here area 1 means vertical movement, 2 means lateral movement, 3 means horizontal movement, and 4 means landing the quadcopter.

### 6. CONCLUSIONS

We consider the problem of controlling the movement of a quadcopter, the trajectory of which must pass through each of the given points. To solve this problem using the LQR theory, a computational algorithm is proposed, which ensures fairly accurate passage of the trajectory through given points.

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