TWMS J. App. and Eng. Math. V.15, N.2, 2025, pp. 291-297

MACWILLIAMS IDENTITIES OF THE LINEAR CODES OVER $\frac{\mathbb{Z}_4[u,v]}{\langle u^2,v^2,uv,vu\rangle}$

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ABSTRACT. In this paper, complete weight enumerators, the symmetrized weight enumerators and the Lee weight enumerators for the linear codes over the ring $S = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$, where $u^2 = v^2 = uv = vu = 0$ are defined. The MacWilliams identity denotes an identity between a linear code and its dual code on their weight distribution. We classify elements of S into seven classes and study MacWilliams identities of linear codes over S. Finally, we calculate the Lee weights of Gray images of the elements and give an example.

Keywords: Gray map, linear codes, MacWilliams identities, weight enumerators.

AMS Subject Classification: 11T71, 94B05

This paper is a revised and expanded version of the paper entitled "MacWilliams Identities For The Linear Codes Over $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$" presented at the "2nd International Conference on Applied Engineering and Natural Sciences", Konya Technical University, Konya, Türkiye, March 2022.

1. Introduction

MacWilliams identities are related to the weight enumerator of a linear code and the weight enumerator of its dual code. It is one of the most fundamental results in coding theory. In the last few decades, the study of codes over various finite rings has received much attention. A number of papers have been published on MacWilliams identities for linear codes over finite rings for various types of weight enumerators. For example, in [10] linear codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ were investigated and MacWilliams identities for a variety of weight enumerators were proven. In [3] linear codes were studied over the ring $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$, where p is an odd prime. Also, a Gray map and MacWilliams identity were given. In [7] linear codes were considered over the ring $\mathbb{Z}_4[u]/\left\langle u^2-1\right\rangle$ and Lee weights, Gray maps and all weight enumerators for these codes were defined and MacWilliams identities for the complete, symmetrized and Lee weight enumerators were proved. In [8] the authors investigated linear codes over the ring $\mathbb{Z}_4+u\mathbb{Z}_4+u^2\mathbb{Z}_4$. Lee weights, Gray maps

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§ Manuscript received: August 12, 2023; accepted: January 13, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.2;

for these codes were defined and MacWilliams identities for the complete, symmetrized and Lee weight enumerators were proven.

Extensions of the ring $\mathbb{Z}_4$ received a special attention in the study of codes over rings. For example, in [6] linear codes over $R = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4 + uv\mathbb{Z}_4$ and MacWilliams identities for linear codes over R with respect to both Lee and Hamming weight enumerators were obtained. In a recent work [5] the authors considered the commutative ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$, where $u^2 = v^2 = uv = vu = 0$ with 64 elements. They completely determined the generators of cyclic and constacyclic codes over this ring. Moreover, they constructed the minimal generating sets for cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$.

Motivated by these works, we define complete weight enumerators, the symmetrized weight enumerators and the Lee weight enumerators for linear codes over the ring $S = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$, where $u^2 = v^2 = uv = vu = 0$ and prove MacWilliams identities for all the weight enumerators involved.

2. Preliminaries

Consider the ring $S = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$, where $u^2 = v^2 = uv = vu = 0$. It can be also viewed as the quotient ring $\mathbb{Z}_4[u,v]/\langle u^2, v^2, uv, vu\rangle$. Any element of $s \in S$ can be expressed uniquely as $s = a + ub + vc$, where $a, b, c \in \mathbb{Z}_4$. The finite ring S has the following properties:

- It has 64 elements.
- Its units are given by

$$U = \{a^* + ub + vc \ : \ a^* \text{ is unit in } \mathbb{Z}_4, b, c \in \mathbb{Z}_4\}.$$

- It is a local Frobenius ring with unique maximal ideal $I = \langle 2, u, v\rangle$.
- It is a non-chain extension of the ring $\mathbb{Z}_4$.
- It is not a principal ideal ring [5].

Recall that a linear code C of length n over the ring S is an S-submodule of S^n. For any element $\mathbf{s} = (s_0, s_1, \ldots, s_{n-1})$ of S^n, the cyclic shift operator is defined as:

$$\sigma(s_0, s_1, \ldots, s_{n-1}) = (s_{n-1}, s_0, \ldots, s_{n-2}).$$

Let C be a linear code of length n over S, then C is called cyclic if $\sigma(C) = C$.

The Lee weight w_L of any element a of $\mathbb{Z}_4$ as

$$w_L(a) = \min\{a, 4 - a\}.$$

The Lee weight $w_L(\mathbf{a})$ of a vector $\mathbf{a} \in \mathbb{Z}_4^n$ is defined as the rational sum of the Lee weights of its coordinates. In [5] the Gray map was defined as follows

$$\phi : S \to \mathbb{Z}_4^3$$

$$a + ub + vc \mapsto (a, a + b, a + c).$$

We define the Lee weight of any element $s = a + ub + vc \in S$ as $w_L(\phi(s))$. That is,

$$w_L(s) = w_L(a, a + b, a + c)$$

where $a, b, c \in \mathbb{Z}_4$ [5].

This map is extended componentwise to

$$\Phi : S^n \to \mathbb{Z}_4^{3n}$$

and the Lee weight $w_L(\mathbf{s})$ of $\mathbf{s} \in \mathbb{Z}_4^{3n}$ is defined as the rational sum of Lee weights of its coordinates.

Let $\mathbf{w} = (w_0, w_1, \ldots, w_{n-1})$ and $\mathbf{z} = (z_0, z_1, \ldots, z_{n-1}) \in S^n$. The Euclidean inner product of $\mathbf{w}$ and $\mathbf{z}$ is defined as

$$\mathbf{w} \cdot \mathbf{z} = w_0 z_0 + w_1 z_1 + \cdots w_{n-1} z_{n-1}$$

where the operations are performed in the ring S.

Definition 2.1. *Let C be linear code of length n over S. Then the dual of C is defined as*

$$C^{\perp} = \{\mathbf{w} \in S^n : \ \mathbf{w} \cdot \mathbf{v} = 0 \textit{ for all } \mathbf{v} \in C\}.$$

3. Weight Enumerators and MacWilliams Identities

3.1. **The Complete Weight Enumerator.** The complete weight enumerator gives us a lot of information about the code such as the size of the code, the minimum weight of the code and the weight enumerator of the code for any weight function. Since S is a Frobenius ring, the MacWilliams identities for the complete weight enumerator hold.

We list the elements of the ring S as $S = \{g_1, g_2, \ldots, g_{64}\}$ given in Table 1 along with the Gray image of each element. Next, we partition the elements of S into 7 classes based on Lee weights as N_0, N_1, N_2, N_3, N_4, N_5, N_6, where for $0 \le i \le 6$,

$$N_i = \{s \in S : w_L(s) = i\}.$$

The size of each N_i is given as $|N_0| = |N_6| = 1$, $|N_1| = |N_5| = 6$, $|N_2| = |N_4| = 15$, $|N_4| = 20$. Then define

$$N_i N_j = \{xy : x \in N_i, y \in N_j\}.$$

Clearly, $N_0 N_j = N_0$ $(0 \le j \le 6)$.
The complete weight enumerator (cwe) of a linear code C over S is defined by

$$cwe_C(x_1, x_2, \ldots, x_{64}) = \sum_{\mathbf{d} \in C} x_1^{wt_{g_1}(\mathbf{d})} x_2^{wt_{g_2}(\mathbf{d})} \cdots x_{64}^{wt_{g_{64}}(\mathbf{d})}$$

where $wt_{g_i}(\mathbf{d})$ is the number of g_i in the codeword $\mathbf{d}$. This is a homogeneous polynomial in 64 variables $x_1, x_2, \ldots, x_{64}$ with total degree on each term being n, the length of C.

Remark 3.1. *We observe the following basic facts about the cwe of a code.*

1. *Since $0 \in C$, the term x_1^n always appears in $cwe_C(x_1, x_2, \ldots, x_{64})$.*
2. *$cwe_C(1, 1, \ldots, 1) = |C|$.*
3. *$cwe_C(a, 0, \ldots, 0) = a^n$.*

Definition 3.1. *Let I be a non-zero ideal of S. Define $\chi : I \to \mathbb{C}^*$ by $\chi(a + ub + vc) = i^{a+b+c}$, where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers, and χ is a non-trivial character of I. We then define the 64×64 matrix M, by letting $M(i,j) = \chi(g_i g_j)$.*

We have the following theorem from [9].

Theorem 3.1. *Let C be a linear code of length n over S. Then*

$$cwe_{C^{\perp}}(x_1, x_2, \ldots, x_{64}) = \frac{1}{|C|} cwe_C\left(M\left(x_1, x_2, \ldots, x_{64}\right)^T\right)$$

where M is the 64×64 matrix defined by $M(i,j) = \chi(g_i g_j)$ and T represents the transpose.

3.2. **The Symmetrized Complete Weight Enumerator and Lee Weight Enumerator.** Since in $\mathbb{Z}_4$, $w_L(1) = w_L(3) = 1$, the symmetrized weight enumerator the code over $\mathbb{Z}_4$ was defined in [2] as

$$swe_C\left(x, y, z\right) = cwe_C\left(x, y, z, y\right).$$

Adopting the same idea, we will define the symmetrized weight enumerator of the code over S. To do this we need Table 2 which gives us the elements of S, their Lee weights and the corresponding variables. Considering the elements that have the same weights we can define the symmetrized weight enumerator as follows:

Definition 3.2. *Let C be a linear code over S of length n. Then define the symmetrized weight enumerator of C as*

$$swe_C\left(x, y, z, w, p, r, s\right) = cwe_C\left(x, \underbrace{y, \ldots, y}_{6}, \underbrace{z, \ldots, z}_{15}, \underbrace{w, \ldots, w}_{20}, \underbrace{p, \ldots, p}_{15}, \underbrace{r, \ldots, r}_{6}, s\right),$$

where x, y, z, w, p, r, s represent the elements of Lee weight 0,1,2,3,4,5,6 respectively. Therefore,

$$swe_C\left(x, y, z, w, p, r, s\right) = \sum_{\mathbf{d}\in C} x^{wt_0(\mathbf{d})} y^{wt_1(\mathbf{d})} z^{wt_2(\mathbf{d})} w^{wt_3(\mathbf{d})} p^{wt_4(\mathbf{d})} r^{wt_5(\mathbf{d})} s^{wt_6(\mathbf{d})},$$

where $wt_0(\mathbf{d}) = wt_{g_1}(\mathbf{d})$, $wt_1(\mathbf{d}) = \sum_{i=2}^{7} wt_{g_i}(\mathbf{d})$, $wt_2(\mathbf{d}) = \sum_{i=8}^{22} wt_{g_i}(\mathbf{d})$, $wt_3(\mathbf{d}) = \sum_{i=23}^{42} wt_{g_i}(\mathbf{d})$, $wt_4(\mathbf{d}) = \sum_{i=43}^{57} wt_{g_i}(d)$, $wt_5(\mathbf{d}) = \sum_{i=58}^{63} wt_{g_i}(\mathbf{d})$, $wt_6(\mathbf{d}) = wt_{g_{64}}(\mathbf{d})$.

Theorem 3.2. *Let C be a linear code of length n over S. Then*
$swe_{C^\perp}\left(x, y, z, w, p, r, s\right) = \dfrac{1}{|C|} swe_C\left(D_0, D_1, D_2, D_3, D_4, D_5, D_6\right)$, where

$$\begin{aligned}
D_0 &= x + 6y + 15z + 20w + 15p + 6r + s \\
D_1 &= x + 4y + 5z - 5p - 4r - s \\
D_2 &= x + 2y - z - 4w - p + 2r + s \\
D_3 &= x - 3z + 3p - s \\
D_4 &= x + 2y - z - 4w - p + 2r + s \\
D_5 &= x - 4y + 5z - 5p + 4r - s \\
D_6 &= x - 6y + 15z - 20w + 15p - 6r + s
\end{aligned}$$

Proof. The proof is similar to that for Theorem 8 in [6]. It is obvious that if $r \in N_0$, then $\sum\limits_{s\in N_0} \chi(rs) = 1$, $\sum\limits_{s\in N_1} \chi(rs) = 6$, $\sum\limits_{s\in N_2} \chi(rs) = 15$, $\sum\limits_{s\in N_3} \chi(rs) = 20$, $\sum\limits_{s\in N_4} \chi(rs) = 15$, $\sum\limits_{s\in N_5} \chi(rs) = 6$, $\sum\limits_{s\in N_6} \chi(rs) = 1$.

From the proof of Theorem 8, if $w_L(\alpha) = w_L(\beta)$ for $\alpha, \beta \in S$, we have

$$\sum_{s\in N_j} \chi(\alpha s) = \sum_{s\in N_j} \chi(\beta s), (1 \leq j \leq 6).$$

So, if $r \in N_1$, then $\sum\limits_{s\in N_0} \chi(rs) = 1$, $\sum\limits_{s\in N_1} \chi(\alpha g) = 4$, $\sum\limits_{s\in N_2} \chi(rs) = 5$, $\sum\limits_{s\in N_3} \chi(rs) = 0$, $\sum\limits_{s\in N_4} \chi(rs) = -5$, $\sum\limits_{s\in N_5} \chi(rs) = -4$, $\sum\limits_{s\in N_6} \chi(rs) = -1$. Others can be obtained similarly. □

Definition 3.3. [4] *Let C be a linear code of length n over $\mathbb{Z}_4$. Then, the Lee weight enumerator of C is defined by*

$$Lee_C(x,y) = \sum_{c\in C} x^{2n-wt(c)} y^{wt(c)}.$$

Definition 3.4. *Let C be a linear code of length n over S. Then, the Lee weight enumerator of C is defined by*

$$Lee_C(x,y) = \sum_{\mathbf{d}\in C} x^{6n-wt_L(\mathbf{d})} y^{wt_L(\mathbf{d})}.$$

Theorem 3.3. *Let C be a linear code of length n over S. Then*

$$Lee_C(x,y) = swe_C\left(x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6\right).$$

Proof. Let $wt_L(\mathbf{d}) = \sum_{i=0}^{6} i wt_i(\mathbf{d})$. For $n = \sum_{i=1}^{64} wt_{g_i}(\mathbf{d}) = \sum_{i=0}^{6} wt_i(\mathbf{d})$, we have

$$6n - wt_L(\mathbf{d}) = \sum_{i=0}^{6}(6-i)wt_i(\mathbf{d}).$$

From the definition of the Lee weight enumerator of C above, we have

$$\begin{aligned} Lee_C(x,y) &= \sum_{\mathbf{d}\in C} x^{6n-wt_L(\mathbf{d})} y^{wt_L(\mathbf{d})} \\ &= \sum_{\mathbf{d}\in C} x^{\sum_{i=0}^{6}(6-i)wt_i(\mathbf{d})} y^{\sum_{i=0}^{6}(6-i)wt_i(\mathbf{d})} \\ &= \sum_{\mathbf{d}\in C} \prod_{i=0}^{6} (x^{6-i}y^i)^{wt_i(\mathbf{d})} \\ &= swe_C\left(x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6\right). \end{aligned}$$

□

Theorem 3.4. *Let C be a linear code of length n over S. Then*

$$Lee_{C^\perp}(x,y) = \frac{1}{|C|} Lee_C(x+y, x-y).$$

Proof. From Theorems 3.2 and 3.3, we have

$$Lee_{C^\perp}(x,y) = \frac{1}{|C|} swe_C(E_0, E_1, E_2, E_3, E_4, E_5, E_6)$$

where

$$\begin{aligned} E_0 &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 = (x+y)^6 \\ E_1 &= x^6 + 4x^5y + 5x^4y^2 - 5x^2y^4 - 4xy^5 - y^6 = (x+y)^5(x-y) \\ E_2 &= x^6 + 2x^5y - x^4y^2 - 4x^3y^3 - x^2y^4 + 2xy^5 + y^6 = (x+y)^4(x-y)^2 \\ E_3 &= x^6 - 3x^4y^2 + 3x^2y^4 - y^6 = (x+y)^3(x-y)^3 \\ E_4 &= x^6 + 2x^5y - x^4y^2 - 4x^3y^3 - x^2y^4 + xy^5 + y^6 = (x+y)^2(x-y)^4 \\ E_5 &= x^6 - 4x^5y + 5x^4y^2 - 5x^2y^4 + 4xy^5 - y^6 = (x+y)(x-y)^5 \\ E_6 &= x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6 = (x-y)^6 \end{aligned}$$

Hence $Lee_{C^\perp}(x,y) = \frac{1}{|C|} Lee_C(x+y, x-y)$. □

Example 3.1. *Let C be the linear code of length 3 over S generated by*

$$G = \begin{pmatrix} 3+2u+2u & 0 & 0 & 1+2u & 0 & 0 \\ 0 & 3+2u+2u & 0 & 0 & 1+2u & 0 \\ 0 & 0 & 3+2u+2u & 0 & 0 & 1+2u \end{pmatrix}.$$

By using Magma [1] *we obtain the Lee weight enumerator of C as* $x^{36}+6x^{30}y^{6}+15x^{24}y^{12}+20x^{18}y^{18}+15x^{12}y^{24}+6x^{6}y^{30}+y^{36}$. *The dual code $C^{\perp}$ is generated by*

$$H = \begin{pmatrix} 1+2u+2v & 0 & 0 & 1+2u & 0 & 0 \\ 0 & 1+2u+2v & 0 & 0 & 1+2u & 0 \\ 0 & 0 & 1+2u+2v & 0 & 0 & 1+2u \end{pmatrix}.$$

Moreover we have $Lee_{C^{\perp}}(x, y) = \frac{1}{|C|} Lee_{C}(x+y, x-y)$.

4. Conclusion

In this paper, weight enumerators for the linear codes over the ring $S = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$, where $u^2 = v^2 = uv = vu = 0$ are studied. MacWilliams identities for the complete, symmetrized and Lee weight enumerators are proved.

Acknowledgement. We thank the referees for useful suggestions to improve the presentation of this paper.

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TABLE 1. Lee weights of the elements of S

g_i $1 \leq i \leq 64$	The Gray image of g_i	Lee weight of g_i	The corresponding variable
0	000	0	g_1
v	001	1	g_2
3v	003	1	g_3
u	010	1	g_4
3u	030	1	g_5
1+3u+3v	100	1	g_6
3+u+v	300	1	g_7
2v	002	2	g_8
u+v	011	2	g_9
u+3v	013	2	g_{10}
2u	020	2	g_{11}
3u+v	031	2	g_{12}
3u+3v	033	2	g_{13}
1+3v	110	2	g_{14}
1+2u+3v	130	2	g_{15}
1+3u	101	2	g_{16}
1+3u+2v	103	2	g_{17}
2+2u+2v	200	2	g_{18}
3+v	330	2	g_{19}
3+u	303	2	g_{20}
3+2u+v	310	2	g_{21}
3+u+2v	301	2	g_{22}
u+2v	012	3	g_{23}
2u+v	021	3	g_{24}
2u+3v	023	3	g_{25}
3u+2v	032	3	g_{26}
1	111	3	g_{27}
1+2v	113	3	g_{28}
1+u+3v	120	3	g_{29}
1+3u+v	102	3	g_{30}
1+2u+2v	133	3	g_{31}
2+u+2v	230	3	g_{32}
2+2u+v	203	3	g_{33}
2+2u+3v	201	3	g_{34}
2+3u+2v	210	3	g_{35}
3	333	3	g_{36}
3+2v	331	3	g_{37}
3+u+3v	302	3	g_{38}
3+2u	313	3	g_{39}
3+2u+2v	311	3	g_{40}
3+3u+v	320	3	g_{41}
1+2u	131	3	g_{42}
1+v	112	4	g_{43}
2u+2v	022	4	g_{44}
1+u	121	4	g_{45}
1+u+2v	123	4	g_{46}
1+2u+v	132	4	g_{47}
2+2v	220	4	g_{48}
2+u+v	233	4	g_{49}
2+u+3v	231	4	g_{50}
2+2u	202	4	g_{51}
2+3u+v	213	4	g_{52}
2+3u+3v	211	4	g_{53}
3+3v	332	4	g_{54}
3+2u+3v	312	4	g_{55}
3+3u	323	4	g_{56}
3+3u+2v	321	4	g_{57}
1+u+v	122	5	g_{58}
2+v	223	5	g_{59}
2+3v	221	5	g_{60}
2+u	232	5	g_{61}
2+3u	212	5	g_{62}
3+3u+3v	322	5	g_{63}
2	222	6	g_{64}