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A NUMERICAL SOLUTION OF THE MATHEMATICAL MODELS FOR WATER POLLUTION BY SHIFTED JACOBI POLYNOMIALS

A. EBRAHIMZADEH[1*], E. HASHEMIZADEH[2], R. MIRABBASI[3], §

ABSTRACT. Water pollution is one of the most significant environmental issues in developing countries, particularly in relation to drinking water quality. Therefore, monitoring and modeling the quality of water resources is very important in managing the exploitation and protection of water resources. This study presents a numerical approach for solving a mathematical model of soluble and insoluble water pollutants by utilizing shifted Jacobi polynomials (SJP). The transmissibility of water pollution was investigated using a system of ordinary differential equations. In this essay, a nonlinear system of ordinary differential equations is turned into an algebraic system by utilizing the collocation approach based on SJP. Finally, the Newton's method is used to obtain numerical experiments. We also compared present method results by Runge-Kutta (RK) method to demonstrate the efficiency of the propounded method, which shows the results obtained are acceptable and in good agreement with the RK method.

Keywords: Water pollutants, Collocation method, Operational matrix of derivatives, Mathematical model, Shifted Jacobi polynomials.

AMS Subject Classification: 65L60, 91B76

1. Introduction

The accessibility of clean drinking water is one of the challenges facing human societies. Unfortunately, water pollutants in rivers and coastal areas pose a serious threat to aquatic ecosystems in many parts of the world [18]. Water pollutants are substances that contaminate water and change its physical, chemical, or biological properties [12]. These substances can come from a variety of sources, including agricultural fertilizers and pesticides, industrial waste, sewage treatment plants, and stormwater runoff [12]. The foundation of pollutants happens when nutrients, pathogens, plastics, and chemicals such

[1]Department of Mathematics Education, Farhangian University, P.O.Box 14665-889, Tehran, Iran.
e-mail: a.ebrahimzadeh@cfu.ac.ir; https://orcid.org/0000-0002-4684-7640.
[*] Corresponding author.
[2] Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
e-mail: hashemizadeh@kiau.ac.ir; https://orcid.org/0000-0001-5902-4830.
[3] Department of Water Engineering, Faculty of Agriculture, Shahrekord University, Shahrekord, Iran.
e-mail: mirabbasi@sku.ac.ir; https://orcid.org/0000-0002-9897-0042.
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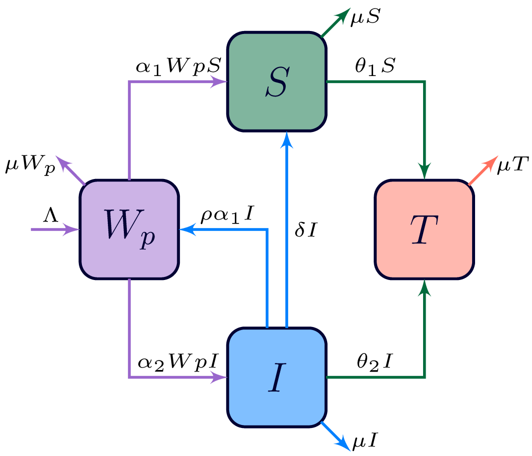


FIGURE 1. The model diagram for the transmission of water pollutants

as antibiotics, heavy metals, and pesticides are discharged into water bodies like oceans, lakes, rivers, and groundwater [2].

Water pollution can be caused by a wide variety of substances including pathogenic microorganisms, putrescence organic waste, fertilizers and plant nutrients, toxic chemicals, sediments, heat, petroleum (oil), and radioactive substances [2]. The main causes of water pollution are attributed to industrial activities, urbanization, religious and social practices, agricultural fertilizers and pesticides, and accidents such as oil spills and nuclear fallout [17]. There are several methods for controlling water pollution including physical methods such as sedimentation and filtration, chemical methods such as coagulation and disinfection, biological methods such as activated sludge treatment and constructed wetlands, and ecological methods such as riparian buffer zones [17].

Detailed discussions related to the definitions of water pollution, various kinds of water pollution and its causes and control methods can be seen in the reference [12]. A mathematical model is a simplified representation of a real-world system that uses mathematical language to describe the relationships between different variables [2]. Recently, the mathematical models have been utilized repeatedly to describe the dynamics of infectious diseases such as HIV, HBV, Ebola, H1N1, cancer, malaria, COVID-19, and etc [13, 20, 21, 3, 36, 33]. This essay presents a numerical solution for solving a mathematical model in [2] which describes the relationships between different variables that affect the transmission of water pollutants. The dynamical system for the transmission of water pollutants is described as follows:

$$\begin{aligned} W' &= \Lambda - \alpha_1\ W\,S - \alpha_2\ W\,I + \rho\alpha_2 I - \mu\,W, \quad W \geq 0,\\ S' &= \alpha_1 W\,S + \delta\ I - (\theta_1 + \mu)S, \quad S \geq 0,\\ I' &= \alpha_2 W\,I - \rho\alpha_2\ I - (\delta + \theta_2 + \mu)I, \quad I \geq 0,\\ T' &= \theta_1 S + \theta_2 I - \mu\,T, \quad T \geq 0, \end{aligned} \tag{1}$$

with initial conditions

$$W(0) = W_0, \quad S(0) = S_0, \quad I(0) = I_0, \quad T(0) = T_0. \tag{2}$$

In this paper, we want to explore the transmission of water pollutants by determining the unknowns W, S, I, and T, where W represents the concentration of water pollution in parts per million (PPM), S is the volume of solvable water pollutants, and I is the volume of insolvable water pollutants. T shows the volume of insoluble water that is removed by treatment. The diagram that shows the transmission of water pollutants can be found in Figure 1.

In water science and engineering, it is essential to calculate these variables to determine the progression of water pollutant transfer. The models can provide valuable information to designers and planners for the implementation of water treatment or control plans. Table 1 provides the definition of parameters. To investigate the feasibility and positivity of solutions in model (1), one can refer to [2]. In the dynamics of a system, it is best to classify equilibrium points based on their stability according to the Routh Hurwitz criteria in [26]. The global and local stability of system (1) are discussed in [2] based on the equilibrium point. The sensitivity analysis is also verified in [2]. Sensitivity analysis needs to know how decisive, the role of each parameter is in the transmission of water pollutants [6, 25].

Table 1. Definitions of parameters in the transmission of water pollutant model

Parameters	Descriptions
Λ	The rate of water pollutant
α_1	The transmission rate for soluble water pollutant
α_2	The transmission rate for insoluble water pollutant
ρ	The rate of insoluble water pollutants becoming water pollution
μ	The removable rate of water pollutants
δ	The amount of treating insoluble water pollutants to solute
θ_1	The amount of solvable water pollutants treated
θ_2	The amount of insolvable water pollutants treated

In recent years, various researchers have created different models for water pollution transmission and presented numerical methods to solve them. Agusto and Bamigbola in [1] dissolve the mathematical models of water pollution with the Crank-Nicolson numerical approach. The ultimate results demonstrate that contaminant concentration diminished faster hereon. The authors in [24] presented a model for the dispersion of river pollution using the density of pollutant variables and dissolved it with the finite element method. Their approach resulted in the concentration of water pollutants being reduced to the standard specified by the world health organization. A mathematical model for the transmission of water pollutants is constructed by Parsaie and Haghabi [22] in 2015. They have used the finite volume method and an artificial neural network for soft computing techniques to solve it. Their simulation results showed that their model is suitable for predicting the longitudinal dispersion coefficient of pollution in the Severn River. Shah et al. in [31] constructed a non-linear mathematical model for the transmission of water pollutants. They applied control to insoluble water pollutants to transform them into soluble water pollutants. The stability of the model is verified according to the basic reproduction number. By utilizing proper treatment, the river pollutant is removed to some extent. We have developed a mathematical model for river pollution that consists of a pair of nonlinear equations that are coupled together. We have also studied how the degradation of pollutants is affected by aeration. In 2009, Pimpuncha et al. [23] studied how aeration affects the degradation of pollutants. The mathematical model consists of two advection-dispersion equations that are coupled together.

In this paper, a collocation approach based on SJP and its operational matrix of derivative is utilized to convert the mathematical model of water pollutant into an algebraic system. Finally, the Newton method has been utilized to obtain the numerical solution. This method can be applied to a wider range of real-life mathematical models after minor revision, particularly those with more complicated systems and geometry. Also, implementing the derivative operational matrices of SJP is easy. According to (19), these matrices have many zero elements and are sparse, which makes the method very efficient. The collocation approach based on SJP is also easy to implement and have good convergence properties. They increase the accuracy of the solution due to the least-squares minimization property of the orthogonal polynomials [34].

The paper is structured as follows: Section 2 presents some fundamental properties of SJPs and their operational matrix. In section 3, we explore the use of SJPs and their operational matrix of derivatives in a collocation approach to solve a system of ordinary differential equations that models the transmission of water pollutants. The numerical results are presented in section 4. Finally, section 5 concludes the paper.

2. Some basic properties of SJP

By means of the main properties of Jacobi polynomials, we conclude the following:

$$\mathcal{P}_{k+1}^{(\alpha,\beta)}(t) = \left(a_k^{(\alpha,\beta)} - b_k^{(\alpha,\beta)}\right)\mathcal{P}_k^{(\alpha,\beta)}(t) - c_k^{(\alpha,\beta)}\mathcal{P}_{k-1}^{(\alpha,\beta)}(t), \quad k \geq 1.$$

$$\mathcal{P}_0^{(\alpha,\beta)}(t) = 1, \quad \mathcal{P}_1^{(\alpha,\beta)}(t) = \frac{1}{2}(\alpha+\beta+2)t + \frac{1}{2}(\alpha-\beta),$$

$$\mathcal{P}_k^{(\alpha,\beta)}(-t) = (-1)^k\mathcal{P}_k^{(\alpha,\beta)}(t), \quad \mathcal{P}_k^{(\alpha,\beta)}(-1) = \frac{(-1)^k\Gamma(k+\beta+1)}{k!\Gamma(\beta+1)}, \tag{3}$$

in which $\alpha,\beta > -1, x \in [-1,1]$ and $a_k^{(\alpha,\beta)}$, $a_k^{(\alpha,\beta)}$, and $a_k^{(\alpha,\beta)}$ are defined as follows

$$a_k^{(\alpha,\beta)} = \frac{(2k+\alpha+\beta)(2k+\alpha+\beta+2)}{2(k+1)(k+\alpha+\beta+1)},$$

$$b_k^{(\alpha,\beta)} = \frac{(\beta^2-\alpha^2)(2k+\alpha+\beta+1)}{2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)},$$

$$c_k^{(\alpha,\beta)} = \frac{(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)}{(k+1)(k+\alpha+\beta+1)(2k+\alpha)},$$

For the shifted Jacobi polynomials $\mathcal{P}_{T,k}^{(\alpha,\beta)}(t) = \mathcal{P}_k^{(\alpha,\beta)}(\frac{2t}{T}-1), T>0$, the explicit analytic form is defined as [15, 35]

$$P_{T,i}^{(\alpha,\beta)}(t) = \sum_{j=0}^{k}(-1)^{k-j}\frac{\Gamma(k+\beta+1)\Gamma(j+k+\alpha+\beta+1)}{\Gamma(j+\beta+1)\Gamma(k+\alpha+\beta+1)(k-j)!j!T^j}t^j, \tag{4}$$

$$= \sum_{j=0}^{k}\frac{\Gamma(k+\alpha+1)\Gamma(k+j+\alpha+\beta+1)}{j!(k-j)!\Gamma(j+\alpha+1)\Gamma(k+\alpha+\beta+1)T^j}(t-T)^j. \tag{5}$$

We conclude the following:

$$\mathcal{P}_{T,k}^{(\alpha,\beta)}(0) = (-1)^k\frac{\Gamma(k+\beta+1)}{\Gamma(\beta+1)k!} \tag{6}$$

The set of SJPs $\{P_{T,i}^{(\alpha,\beta)}(t)\}_{i=0}^{n}$ for $i = 0, 1, \cdots$ in $[0,T]$ is defined as follows [7]:

$$P_{T,i}^{(\alpha,\beta)}(t) = \frac{(\alpha+\beta+2i-1)\left\{\alpha^2-\beta^2+\left(\frac{2t}{T}-1\right)(\alpha+\beta+2i)(\alpha+\beta+2i-2)\right\}}{2i\,(\alpha+\beta+i)\,(\alpha+\beta+2i-2)}. \tag{7}$$

The SJPs $P_{T,i}^{(\alpha,\beta)}$ of degree i have an analytic form given by the equation below

$$P_{T,i}^{(\alpha,\beta)}(t) = \sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!k!T^k}t^k, \tag{8}$$

where,

$$P_{T,i}^{(\alpha,\beta)}(0) = (-1)^i\frac{\Gamma(i+\beta+1)}{\Gamma(\beta+1)i!}. \tag{9}$$

They are orthogonal over the interval $[0,T]$ with respect to the weight function $W_T^{(\alpha,\beta)}(t) = t^\beta(T-t)^\alpha$ and satisfy the following properties for $k = 0, 1, \cdots, n$, as follows:

$$\int_0^T P_{T,j}^{(\alpha,\beta)}(t)P_{T,k}^{(\alpha,\beta)}(t)W_T^{(\alpha,\beta)}(t)dt = h_k, \tag{10}$$

in which,

$$h_k = \begin{cases} \frac{T^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\beta+\alpha+1)} & i = j, \\ 0, & i \neq j. \end{cases} \tag{11}$$

A function $f(t) \in L^2(0,1)$ can be expressed in the terms of the SJPs as follows:

$$f(t) = \sum_{k=0}^{n} a_k P_{T,j}^{\alpha,\beta}(t) = A^T\phi(t), \tag{12}$$

where, $A = [a_0, a_1, \cdots, a_n]$ and

$$\phi(t) = [P_{T,0}^{\alpha,\beta}(t), P_{T,1}^{\alpha,\beta}(t), \cdots, P_{T,n}^{\alpha,\beta}(t)]^T. \tag{13}$$

The coefficients a_j for $j = 0, 1, \cdots, n$ can be obtained in the following form,

$$a_j = \frac{1}{h_j}\int_0^T W_T^{(\alpha,\beta)(t)}f(t)P_{T,j}^{(\alpha,\beta)}(t)dt. \tag{14}$$

The definition and lemma below will be required in the section 3 from [13].
Definition 1. The tensor product of vectors $X_{\hat{m}} = [x_i]$ and $Y_{\hat{m}} = [y_i]$ is defined as follows:

$$X \otimes Y = (x_i \times y_i)_{\hat{m}} \tag{15}$$

The tensor product of two matrices $M = [m_{i,j}]$ and $N = [n_{i,j}]$ of order $\hat{m} \times \hat{m}$ is expressed by

$$M \otimes N = (m_{ij} \times y_{ij})_{\hat{m}\times\hat{m}}. \tag{16}$$

For more information about tensor products, one can see [32].
Lemma 1.[13] Let the functions $f(t) = f^T\phi(t)$ and $g(t) = g^T\phi(t)$ belong to $L^2[0,1]$ are expressed by SJPs. Then we have

$$f(t)g(t) = \left(f^T \otimes g^T\right)\phi(t). \tag{17}$$

The interested reader can see [10, 15, 14] for the convergent discussion about SJP approximations.
Theorem 1.[14] For any $u \in \mathcal{D}(\mathcal{A}_{\alpha,\beta}^{\frac{r}{2}})$, $r \in \mathbb{N}$ and $0 \le \mu \le r$,

$$||\pi_n^{\alpha,\beta}u - u||_{\mathcal{D}(\mathcal{A}_{\alpha,\beta}^{\frac{\mu}{2}})} < n^{\mu-r}||u||_{\mathcal{D}(\mathcal{A}_{\alpha,\beta}^{\frac{r}{2}})}. \tag{18}$$

The proof of this theorem, he notation $\pi_n^{\alpha,\beta}$ and $\mathcal{D}(\mathcal{A}_{\alpha,\beta}^{\frac{\mu}{2}})$ are given in [14].

2.1. **SJP operational matrix of derivative.** The derivative of vector $\phi(t)$ in equation (13) can be expressed in matrix form as shown in [9].

$$\frac{d\phi(t)}{dt} = D^{(1)}\phi(t). \tag{19}$$

$D^{(1)} = [d_{ij}]$ is the $(n+1)\times(n+1)$ derivative operational matrix of SJP. This matrix is obtained in [9] and defined as follows:

$$D^{(1)} = (d_{ij}) = \begin{cases} A_1(i,j), & i > j, \\ 0, & \textbf{otherwise,} \end{cases} \tag{20}$$

where,

$$\begin{aligned} A_1(i,j) = {} & \frac{T^{\alpha+\beta}(i+\alpha+\beta+1)(i+\alpha+\beta+2)_j(j+\alpha+2)_{i-j-1}\Gamma(j+\alpha+\beta+1)}{(i-j-1)!\Gamma(2j+\alpha+\beta+1)} \\ & \times {}_3F_2\left(\begin{array}{ccc} -i+1+j, & i+j+\alpha+\beta+2 & j+\alpha+1 \\ j+\alpha+2 & 2j+\alpha+\beta+2 & ;1 \end{array}\right). \end{aligned} \tag{21}$$

The proof of obtaining (21) is given in [9] and for the general definition of a generalized hypergeometric series and special ${}_3F_2$ see [19].

3. Implementation of a numerical approach to the water pollutant problem

As per [5], the four variables of system (1) are normalized in the following way:

$$W_p = \frac{W}{N}, \quad S_p = \frac{S}{N}, \quad I_p = \frac{I}{N}, \quad T_p = \frac{T}{N}, \quad \Lambda_1 = \frac{\Lambda}{N}. \tag{22}$$

where, $N = W + S + I + T$. The main model (1) is converted to the following normalized model:

$$\begin{aligned} W_p' &= \Lambda_1 - \alpha_1 N\ W_p\ S_p - \alpha_2 N\ W_p\ I_p + \rho\alpha_2 I_p - \mu W_p, \quad W_p \geq 0, \\ S_p' &= \alpha_1 N W_p\ S_p + \delta\ I_p - (\theta_1+\mu)S_p, \quad S_p \geq 0, \\ I_p' &= \alpha_2 N W_p\ I_p - \rho\alpha_2\ I_p - (\delta+\theta_2+\mu)I_p, \quad I_p \geq 0, \\ T_p' &= \theta_1 S_p + \theta_2 I_p - \mu T_p, \quad T_p \geq 0, \end{aligned} \tag{23}$$

with initial conditions:

$$W_p(0) = W_{p0}, \quad S_p(0) = S_{p0}, \quad I_p(0) = I_{p0}, \quad T_p(0) = T_{p0}. \tag{24}$$

In this paper, we want to obtain the numerical solution of the mathematical model for transmission of water pollutants (23) with initial conditions in (24) by applying the collocation method based on SJPs. We can express our unknown functions in terms of SJPs as an approximation:

$$W_p = A_1^T\phi(t), \quad I_p = A_2^T\phi(t), \quad S_p = A_3^T\phi(t), \quad T_p = A_4^T\phi(t), \tag{25}$$

where, ϕ is defined in equation (13) and vectors $A_i : i = 1, \cdots 4$ are defined as follows:

$$\begin{aligned} A_1 &= [a_0, \cdots, a_n]^T, & A_2 &= [a_{n+1}, \cdots, a_{2n+1}]^T, \\ A_3 &= [a_{2n+2}, \cdots, a_{3n+2}]^T, & A_4 &= [a_{3n+3}, \cdots, a_{4n+3}]^T. \end{aligned} \tag{26}$$

By utilizing Eqs. (19) and (25), we possess

$$\begin{aligned} W_p'(t) &= A_1^T\phi'(t) = A_1^T D^{(1)}\phi(t) & S_p'(t) &= A_1^T\phi'(t) = A_2^T D^{(1)}\phi(t), \\ I_p'(t) &= A_3^T\phi'(t) = A_3^T D^{(1)}\phi(t), & T_p'(t) &= A_4^T\phi'(t) = A_4^T D^{(1)}\phi(t). \end{aligned} \tag{27}$$

and

$$W_p(t) = A_1^T\phi(t), \qquad S_p(t) = A_2^T\phi(t),$$
$$I_p(t) = A_3^T\phi(t), \qquad T_p(t) = A_4^T\phi(t). \tag{28}$$

By substituting equations (27) and (28) in system dynamics (23), we have

$$A_1^T D^{(1)}\phi(t) = \Lambda_1 - \alpha_1 N\ \left(A_1^T \otimes A_2^T\right)\phi(t) - \alpha_2 N\left(A_1^T \otimes A_3^T\right)\phi(t) + \rho\alpha_2 A_3^T\phi(t) - \mu A_1^T\phi(t).$$
$$A_2^T D^{(1)}\phi(t) = \alpha_1 N\left(A_1^T \otimes A_2^T\right)\phi(t) + \delta\ A_3^T\phi(t) - (\theta_1+\mu)A_2^T\phi(t),$$
$$A_3^T D^{(1)}\phi(t) = \alpha_2 N\left(A_1^T \otimes A_3^T\right)\phi(t) - \rho\alpha_2\ A_3^T\phi(t) - (\delta+\theta_2+\mu)A_3^T\phi(t),$$
$$A_4^T D^{(1)}\phi(t) = \theta_1 A_2^T\phi(t) + \theta_2 A_3^T\phi(t) - \mu A_4^T\phi(t), \tag{29}$$

The main objective of this essay is to determine $(4n+4)$ unknowns of a_i for $i = 0, 1, \cdots, 4n+3$. By substituting the initial conditions in equations (28), we can obtain four linear equations as follows:

$$W_{p0} = A_1^T\phi(0), S_{p0} = A_2^T\phi(0), I_{p0} = A_3^T\phi(0), T_{p0} = A_4^T\phi(0). \tag{30}$$

We can obtain $\phi(0)$ from (9). By replacing a set of (n) points $\tau_i = \frac{2i-1}{2(n+1)}$ for $i = 1, 2, \cdots, n$, we can have

$$A_1^T D^{(1)}\phi(\tau_i) = \Lambda_1 - \alpha_1 N\ \left(A_1^T \otimes A_2^T\right)\phi(\tau_i) - \alpha_2 N\left(A_1^T \otimes A_3^T\right)\phi(\tau_i) + \rho\alpha_2 A_3^T\phi(\tau_i) - \mu A_1^T\phi(\tau_i).$$
$$A_2^T D^{(1)}\phi(\tau_i) = \alpha_1 N\left(A_1^T \otimes A_2^T\right)\phi(\tau_i) + \delta\ A_3^T\phi(\tau_i) - (\theta_1+\mu)A_2^T\phi(\tau_i),$$
$$A_3^T D^{(1)}\phi(\tau_i) = \alpha_2 N\left(A_1^T \otimes A_3^T\right)\phi(\tau_i) - \rho\alpha_2\ A_3^T\phi(\tau_i) - (\delta+\theta_2+\mu)A_3^T\phi(\tau_i),$$
$$A_4^T D^{(1)}\phi(\tau_i) = \theta_1 A_2^T\phi(\tau_i) + \theta_2 A_3^T\phi(\tau_i) - \mu A_4^T\phi(\tau_i), \tag{31}$$

for $i = 1, \cdots, n$, we can solve the unknown vectors of the system of $4n + 4$ equations obtained from (30) and (31) by using Newton's iteration approach as described in [13, 16].

4. Numerical outcomes

This section presents the results of numerical simulations conducted in Mathematica 10.4 software on a personal computer with AMD A6-4400M APU with Radeon™ HD processor and 4GB Memory to investigate the transmission rate of water pollutants. The parameter values and initial conditions used in this study are based on the references [2] and [31] as follows: $\Lambda_1 = 0.0008$, $\rho = 0.25$, $\alpha_2 = 0.02$, $\delta = 0.3$; $\mu = 0.4, \theta_1 = 0.2$, $\theta_2 = 0.5, W_{p0} = 0.5$, $I_{p0} = 0.1$, $S_{p0} = 0.4$, and $T_{p0} = 0$. We numerically examined the factors that contribute to water pollution. Solvable and insolvable water pollutants from chemical industries, health centers, and improper discharge of garbage into water bodies were identified as the major factors causing water pollution. Deadly pathogens are rapidly polluting water supplies, which could lead to disease outbreaks in society. The numerical results have been obtained with $n = 5$, $\alpha = 0$, and $\beta = 0$. The errors E_W, E_S, E_I, and E_T in Table 2 are measures of the difference between the solutions from RK and the presented method divided by the range length function. They are calculated as follows:

$$E_w = \frac{W_{RK} - W_P}{R_w}, \quad E_s = \frac{S_{RK} - S_P}{R_s}, \quad E_I = \frac{I_{RK} - I_P}{R_I}, \quad E_T = \frac{T_{RK} - T_P}{R_T}, \tag{32}$$

where W_{RK}, S_{RK}, I_{RK} and T_{RK} are calculated from RK method, W_p, S_p, I_p and T_p are the solutions from presented method, and R_w, R_s, R_I and R_T are the range length functions. According to the Figure 2, we consider $R_w = 500$, $R_s = 800$, $R_I = 100$, $R_T = 150$. According to table 2, the obtained results for errors E_w, E_s, E_I and E_T demonstrate that the relative errors obtained are all small and reliable, and this indicates the accuracy and efficiency of the proposed method. The outcomes of the suggested procedure are

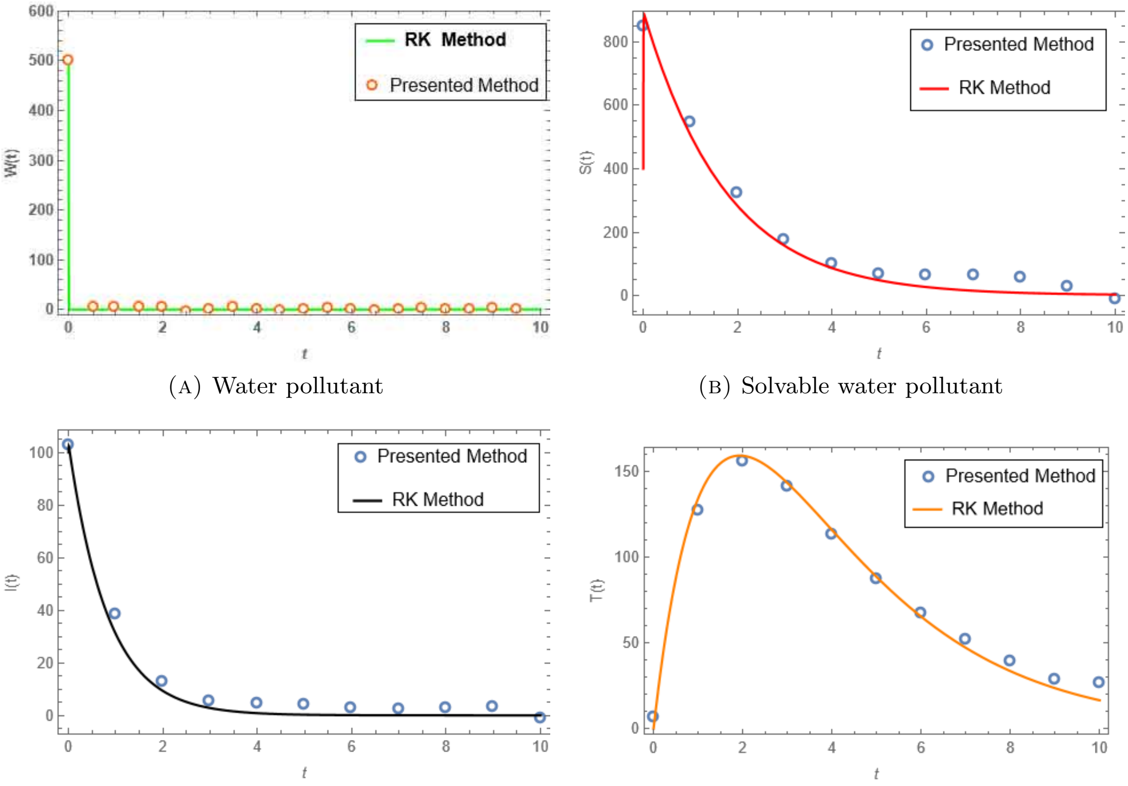
Table 2. Numerical Results for transmission of water pollutant model

t	E_w	E_s	E_I	E_T
0.0	$1.82238E-10$	$5.60724E-01$	$2.74340E-02$	$4.23530E-02$
0.1	$3.92093E-02$	$4.77563E-02$	$7.44891E-02$	$4.87579E-02$
0.2	$9.00948E-03$	$4.84820E-02$	$3.48936E-02$	$2.34431E-02$
0.3	$1.70664E-03$	$2.50430E-02$	$2.56804E-02$	$1.40351E-02$
0.4	$3.77399E-04$	$1.47946E-02$	$3.38708E-02$	$1.65920E-02$
0.5	$2.53874E-04$	$2.49247E-02$	$3.55370E-02$	$9.35553E-03$
0.6	$9.10434E-06$	$4.59110E-02$	$2.70962E-02$	$1.16129E-02$
0.7	$7.35131E-04$	$6.17957E-02$	$2.00486E-02$	$3.24062E-02$
0.8	$3.23355E-03$	$5.81400E-02$	$2.47571E-02$	$3.70840E-02$
0.9	$2.95903E-03$	$2.85325E-02$	$3.09439E-02$	$3.08061E-02$
1.0	$1.41252E-01$	$1.97474E-02$	$1.27969E-02$	$6.63999E-02$

compared with those of RK Method. Figures 2 illustrate the changes in water pollutant variables over time for RK and the presented method in 3. Additionally, Figure 2 demonstrates the convergence behavior of the water pollutant variables for $n = 5$, $\alpha = 0$, and $\beta = 0$. Specifically, Figure 2 (a) shows that the concentration of water pollutants remains relatively stable as the number of days increases. At the beginning of the time period, solvable water containment increased from $400mg/l$ to $900mg/l$ and then decreased as the number of months increased. Additionally, Figure 2 (a) demonstrates that pollutants can be removed from water supplies by implementing treatment. In Figure 2 (b), the sharp increase in soluble water contaminants is expected due to interaction with water pollutants and the transport of insoluble water pollutants to solutes. However, soluble water containment decreased due to treatment. Treatment resulted in a decrease in insoluble water pollutants, as demonstrated in Figure 2 (c). Figures 2 (a), (b) and (c) demonstrate that improved water pollution treatment can lead to a reduction in water pollutants. This reduction in pollutants can decrease their harmful effects on the environment and living organisms. Figure 2 (d) also shows that the water refinery improved for two months due to the high concentration of pollution in the first months, then eliminated it after that. Although our results are good and have an acceptable accuracy, the functions $W(t)$, $S(t)$, and $I(t)$ are discontinuous at a point near zero. This paper utilized continuous functions to estimate the solutions. Therefore, numerical solutions will create Gibbs oscillations [28] that destroy the expected exponential convergence by Jacobian bases. Our suggestion for improving numerical results is to determine the location of discontinuity by discontinuity detection methods and then obtain numerical solutions in separate and continuous intervals. Some other of these methods are in [28, 29, 30] and references therein. These suggestions will be investigated in future works.

5. Conclusion

Water is a main resource for the survival of humanity. Water pollutants can have a significant impact on aquatic ecosystems and human health. Treatment can be used to remove pollutants from the water. In this study, a numerical approach is presented for a mathematical model of the transmission of water pollutants formulated in a system of ordinary differential equations. By using shifted Jacobi polynomials and its operational matrix of derivatives, we convert this mathematical model to an algebraic model. The



(A) Water pollutant (B) Solvable water pollutant

(C) Insolvable water pollutant (D) Treated water pollutant

FIGURE 2. Transmission of the concentration of water pollutant.

results of the proposed method were compared with those of the RK method, which confirmed the results of the present approach.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Asiyeh Ebrahimzadeh is an Assistant Professor in Applied Mathematics in Numerical Analysis at Farhangian University, Iran. She earned her Ph.D. degree in 2015 from the Iran University of Science and Technology where she was the top student. She also obtained her MSc degree from Khajeh Nasir Toosi University of Technology and was accepted as a Ph.D. candidate at Iran University of Science and Technology through exceptional quota. Her research interests include Optimal Control, Integral Equations, Mathematics Programming, Mathematica, Numerical Methods in real-life models, Differential Equations, Fractional Differential Equations, and more.

Elham Hashemizadeh obtained her Ph.D. degree in Applied Mathematics in Numerical Analysis area in year 2012. She was top student in school and university. She ranked number one in M.Sc. and Ph.D. course in Islamic Azad University Karaj Branch, also she selected as top researcher several times. She is now Associate Professor in Islamic Azad University Karaj Branch. Her research interests include Applied mathematics, Integral Equations, Mathematics programming, Mathematica, Numerical Methods, Numerical Algorithms, Differential equations and etc.

Rasoul Mirabbasi is an Associate Professor of Hydrology and Water Resources Engineering at Shahrekord University, Iran. He is also head of the Water Resources Center of Shahrekord University. His research focuses mainly on Statistical and Environmental Hydrology and Climate Change. In particular, he is working on modeling natural hazards including flood, drought, wind, and pollution toward a sustainable environment. Formerly, he was a visiting researcher at the University of Connecticut. He has contributed to more than 150 publications in journals, books, or technical reports. Mirabbasi is the reviewer of about 35 Web of Science (ISI) journals. His Google Scholar citations are 2313 and h-index of 25.