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REFLEXIVE EDGE STRENGTH OF NONAGON CHAIN GRAPH AND TRIANGLE RIBBON LADDER GRAPH

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Abstract. Let G be an undirected, simple and connected graph with vertex set $V(G)$ and egde set $E(G)$. An edge irregular reflexive k-labeling f is labeling such that edges labeled with integers number $1, 2, ..., k_e$ and vertices labeled with even integers $0, 2, ..., 2k_v$, where $k = max\{k_e, 2k_v\}$ of a graph G such that the weights for all edge are distinct. The weight of edge xy in G, denoted by $wt(xy)$ is defined as the sum of edge label and all vertices labels that are incident to that edge. The reflexive edge strength of a graph G which is denoted by $res(G)$ is the minimum value k of the largest label on a graph G that can be labeled with edge irregular reflexive k-labeling. This article will review the k edge irregular reflexive labeling on the nonagon chain graph $C(N_r)$ for $r \geq 2$ and a triangular ribbon ladder graph LSP_n for $n \geq 2$ and determines the strength of the reflexive edges on the graph. The results of these graphs are $res(C(N_r))= \lceil \frac{9r}{3} \rceil$, for $9r \not\equiv 2, 3 \pmod 6$, and $\lceil \frac{9r}{3} \rceil + 1$, for $9r \equiv 3 \pmod 6$. and $res(LSP_n)= \lceil \frac{6n-4}{3} \rceil$, for $6n - 4 \not\equiv 2, 3 \pmod 6$, and $\lceil \frac{6n-4}{3} \rceil + 1$, for $6n - 4 \equiv 2 \pmod 6$.

Keywords: Nonagon chain graph, triangle ribbon ladder graph, edge irregular reflexive labeling, reflexive edge strength.

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1. Introduction

Graph theory is a branch of applied mathematics that specifically studies it about graphics. Graph G is a finite non-empty set $V(G) = \{v_1, v_2, \ldots, v_n\}$ which is called the set of points and the set $E(G) = \{e_1, e_2, \ldots, e_n\}$ is an unordered set of $V(G)$ member pairs called edge sets. In graph theory, many concepts are studied in the application of determination of problem solving. One of which is labeling concept. According to Wallis [1], graph labeling is mapping takes a graph element as the domain to a positive or non-negative integer as codomain. Types of graph labeling are divided into point labeling, edge labeling, and point and edge labeling or called total labeling (Gallian [2]).

Total irregular labeling is divided into edge irregular labeling, point irregular labeling

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and total irregular labeling. In 2017, Ryan *et al* in Bača *et al.* [3] introduced a new concept of totally irregular labeling namely edge irregular reflexive labeling and point irregular reflexive labeling. The total k-labeling of the irregular reflexive edge of a graph G is that labeling takes positive integers from 1 to k as edge labels and retrieves even numbers from 0 to k as point labels such that the weights are on each edge of the graph G is different. Bača *et al.* [4] also defines the edge irregular reflexive strength of the graph G denoted by $res(G)$ is the k minimum value of the largest label. The following is given an entry to specify $res(G)$ according to Ryan *et al* in Bača *et al.*[3]

Lemma 1.1. *For every graph G,*

$$res(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{if } |E(G)| \not\equiv 2,3 \pmod 6, \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{if } |E(G)| \equiv 2,3 \pmod 6. \end{cases}$$

There are several researchs that have been studied for labeling reflexive disorders edge and generates $res(G)$ including the fan graph F_n for $n \geq 3$, graph path P_n, star graph $K_{(1,n)}$, and tadpole graph $T_{(m,n)}$. In 2019, Bača *et al.* [4] examined the cycle graph C_n, and in 2020 Indriati *et al.* [5] investigated the path graph corona complete graph K_1 and corona path graph P_2. Different with Fauziah's *et al.* [7] research that they discuss about nonagon chain graph for $tes(G)$ and Atmadja's *et al.* [8] research on triangle ribbon ladder graph with harmonic labelling, in this research we determain the same graphs for edge irregular reflexive labeling. In this research, we determined $res(G)$ for a nonagon chain graph $C(N_r)$ with $r \geq 2$ and a triangle ribbon ladder graph LSP_n with $n \geq 2$.

2. Research Methods

The research method used in this study is a literature review. The references are in the form of journals, books or writings regarding the irregular reflexive edge of total k-labeling. From this way, we can determine $res(G)$ for the nonagon chain graph $C(N_r)$ and triangle ribbon ladder graph LSP_n.

The steps taken in this research are :

(1) Determine the lower bound $res(G)$ of the nonagon chain graph $C(N_r)$ with $r \geq 2$ and triangle ribbon ladder graph LSP_n with $n \geq 2$ based on Lemma 1.1.
(2) Labeling the nonagon chain graph $C(N_r)$ with $r \geq 2$ and triangle ribbon ladder graph LSP_n with $n \geq 2$ satisfying the lower bound has been determined.
(3) Calculate the weight of each edge of the nonagon chain graph $C(N_r)$ with $r \geq 2$ and triangle ribbon ladder graph LSP_n with $n \geq 2$ using formula $wt(xy)$ is the sum of edge xy label and all point x, y labels that are incident to edge xy.
(4) Finding the general pattern $res(G)$ from the nonagon chain graph $C(N_r)$ with $r \geq 2$ and triangle ribbon ladder graph LSP_n with $n \geq 2$.

3. Result and Discussion

3.1. **Nonagon Chain Graph.** Barrientos [6] defines a chained graph as a graph consisting of blocks $B_1, B_2, B_3, \ldots, B_k$ with $k \geq 2$, so that for every $i, 1 \leq i \leq m-1$, B_i and B_{i+1} intersect at exactly one point so the block intersection graph is a path graph. A chain graph is called a nonagon chain graph, symbolized $C(N_r)$, $r \geq 2$ if every cycle in the form of C_9 and every two cycles has at most one cut point. Nonagon chain graph $C(N_r)$ has $V(C(N_r)) = \{a_i, b_i, c_i, d_i, f_i, g_i, h_i, j_i, a_{i+1}\}$, $E(C(N_r)) = \{a_ib_i, b_ic_i, c_id_i, d_ia_{i+1}, a_if_i, f_ig_i, g_ih_i, h_ij_i, j_ia_{i+1}\}$ for $i = 1, 2, 3, ...r$. The number of points in a nonagon chain graph $C(N_r)$ is $\mid V(G) \mid = 8r + 1$, and the number of sides is $\mid E(G) \mid = 9r$, for $r \geq 2$ (Fauziah [7]).

The strength of the reflexive edge on the nonagon chain graph $C(N_r)$ can be obtained as follows

Theorem 3.1. *For all positive integers $r \geq 2$*

$$res(C(N_r)) = \begin{cases} \lceil \frac{9r}{3} \rceil, & \textit{for } 9r \not\equiv 2,3 \pmod 6, \\ \lceil \frac{9r}{3} \rceil + 1, & \textit{for } 9r \equiv 3 \pmod 6. \end{cases}$$

The lower bound in Theorem 1 is the same as the lower bound of $res(C(N_r))$ shown by Ryan *et al.* [3,4]. The number of edges of this graph is $9r$. Depent on Lemma 1.1, the $res(C(N_r)) \geq \lceil \frac{|E(G)|}{3} \rceil$, for $|E(G)| \not\equiv 2,3 \pmod 6$, and $\lceil \frac{|E(G)|}{3} \rceil + 1$, for $|E(G)| \equiv 3 \pmod 6$. This formula is equivalent with $res(C(N_r)) \geq \lceil \frac{9r}{3} \rceil$, for $9r \not\equiv 2,3 \pmod 6$, and $\lceil \frac{9r}{3} \rceil + 1$, for $9r \equiv 3 \pmod 6$.

Proof. we prove the upper bound on the nonagon chain graph $(C(N_r))$. Build the function f on the nonagon chain graph of $k-$labeling $C(N_r)$ with $k = \lceil \frac{9r}{3} \rceil$, for $9r \not\equiv 2,3 \pmod 6$, and $\lceil \frac{9r}{3} \rceil + 1$, for $9r \equiv 3 \pmod 6$.

$$f(a_i) = \begin{cases} 3i-3, \ i \text{ odd } 1 \leq i \leq r+1, \\ 3i-2, \ i \text{ even } 1 \leq i \leq r+1. \end{cases}$$

$$f(b_i) = f(f_i) = \begin{cases} 3i-3, \ i \text{ odd } 1 \leq i \leq r, \\ 3i-2, \ i \text{ even } 1 \leq i \leq r. \end{cases}$$

$$f(c_i) = f(d_i) = f(g_i) = f(h_i) = f(j_i) = \begin{cases} 3i-1, \ i \text{ odd } 1 \leq i \leq r, \\ 3i, \ i \text{ even } 1 \leq i \leq r. \end{cases}$$

$$f(a_ib_i) = f(b_ic_i) = f(c_id_i) = \begin{cases} 3i-2, \ i \text{ odd } 1 \leq i \leq r, \\ 3i-4, \ i \text{ even } 1 \leq i \leq r. \end{cases}$$

$$f(a_if_i) = f(f_ig_i) = f(g_ih_i) = \begin{cases} 3i-1, \ i \text{ odd } 1 \leq i \leq r, \\ 3i-3, \ i \text{ even } 1 \leq i \leq r. \end{cases}$$

$$f(a_{i+1}d_i) = 3i-2, \ 1 \leq i \leq r.$$

$$f(a_{i+1}j_i) = \begin{cases} 1, \ i = 1, \\ 3i, \ 2 \leq i \leq r. \end{cases}$$

$$f(h_ij_i) = \begin{cases} 3i+1, \ i \text{ odd } 1 \leq i \leq r, \\ 3i-1, \ i \textbf{ even } 1 \leq i \leq r. \end{cases}$$

Based on the proof of the upper and lower bounds of the nonagon chain graph $(C(N_r))$, the maximum value of point labels which are even integers and edge labels which are positive integers is $\lceil \frac{9r}{3} \rceil$, for $9r \not\equiv 2,3 \pmod 6$, and $\lceil \frac{9r}{3} \rceil + 1$, for $9r \equiv 3 \pmod 6$.

Therefore, the edge weights are obtained as follows,

$$w_t(a_ib_i) = 9i-8, \ 1 \leq i \leq r.$$

$$w_t(b_ic_i) = 9i-6, \ 1 \leq i \leq r.$$

$$w_t(c_id_i) = 9i-4, \ 1 \leq i \leq r.$$

$$w_t(d_ia_{i+1}) = 9i-2, \ 1 \leq i \leq r.$$

$$w_t(a_if_i) = 9i-7, \ 1 \leq i \leq r.$$

$$w_t(f_ig_i) = 9i-5, \ 1 \leq i \leq r.$$

$$w_t(g_ih_i) = 9i - 3,\ 1 \le i \le r.$$
$$w_t(h_ij_i) = 9i - 1,\ 1 \le i \le r.$$
$$w_t(j_ia_{i+1}) = 9i,\ 1 \le i \le r.$$

From the research that has been done, it appears that all the edge weights of the nonagon chain graph $(C(N_r))$ are different, the lower and upper bounds are the same as $res(C(N_r))$. Thus f satisfies the element of edge irregular reflexive labeling and has the strength $res(C(N_r))$ according to Theorem 1. Thus the theorem is proven. □

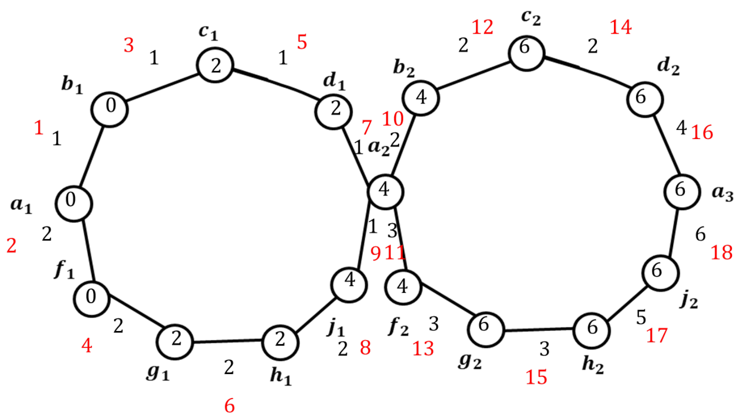

FIGURE 1. Edge irregular reflexive 6-labeling of nonagon chain graph $C(N_2)$

Figure 1 is an illustration of edge irregular reflexive 6-labeling of nonagon chain graph $(C(N_2))$. In Figure 1, the blue numbers are the edge weights of the nonagon chain graph $(C(N_2))$ and the red numbers are the edge labels and point labels of the nonagon chain graph $(C(N_2))$, while the black letters are the names of the point and edge of the nonagon chain $(C(N_2))$.

3.2. **Triangle Ribbon Ladder Graph.** Triangle ribbon ladder graph (LSP_n) is a graph with a set of points $V(G) = \{u_i, v_i \mid 1 \le i \le n\} \cup \{w_i, w_{i+1} \mid 1 \le i \le n-1\}$ and set of edges $E(G) = \{w_iu_i \mid 1 \le i \le n\} \cup \{w_iv_i \mid 1 \le i \le n\} \cup \{w_iw_{i+1}, u_iw_{i+1}, v_iw_{i+1} \mid 1 \le i \le n-1\} \cup \{v_{2i-1}v_{2i}, u_{2i}u_{2i+1} \mid 1 \le i \le n-1\}$. Triangle ribbon ladder graph (LSP_n) is a modified graph of the triangular ladder graph (LS_n) with a shape like an elongated ribbon where the number of points is $3n$, and the number of edges is $6n-4$, with $n \ge 2$ (Atmadja [8]).
The strength of the reflexive edge on triangle ribbon ladder graph (LSP_n) can be obtained as follows

Theorem 3.2. *For all positive integers $n \ge 2$*

$$res(LSP_n) = \begin{cases} \lceil \frac{6n-4}{3} \rceil, & \text{for } 6n-4 \not\equiv 2,3 \pmod 6, \\ \lceil \frac{6n-4}{3} \rceil + 1, & \text{for } 6n-4 \equiv 2 \pmod 6. \end{cases}$$

The lower bound in Theorem 2 is the same as the lower bound of $res(LSP_n)$ shown by Ryan *et al.* [3,4]. The number of edges of this graph is $6n-4$. Depent on Lemma 1.1, the $res(LSP_n) \ge \lceil \frac{|E(G)|}{3} \rceil$, for $|E(G)| \not\equiv 2,3 \pmod 6$, and $\lceil \frac{|E(G)|}{3} \rceil + 1$, for $|E(G)| \equiv 2 \pmod 6$. This formula is equivalent with $res(LSP_n) \ge \lceil \frac{6n-4}{3} \rceil$, for $6n-4 \not\equiv 2,3 \pmod 6$, and $\lceil \frac{6n-4}{3} \rceil + 1$, for $6n-4 \equiv 2 \pmod 6$.

Proof. we prove the upper bound of triangle ribbon ladder graph (LSP_n). Build the function f on triangle ribbon ladder graph of $k-$labeling (LSP_n) with $k = \lceil \frac{6n-4}{3} \rceil$, for $6n - 4 \not\equiv 2, 3 \pmod 6$, and $\lceil \frac{6n-4}{3} \rceil + 1$, for $6n - 4 \equiv 2 \pmod 6$.

$$f(u_i) = \begin{cases} 0, \ i{=}1, \\ 2i, \ 2 \le i \le n. \end{cases}$$

$$f(w_i) = \begin{cases} 0, \ i{=}1{,}2, \\ 2i-2, \ 3 \le i \le n. \end{cases}$$

$$f(v_i) = \begin{cases} 2, \ i{=}1{,}2, \\ 2i, \ 3 \le i \le n. \end{cases}$$

$$f(u_iw_i) = \begin{cases} 1, \ i{=}1, \\ 4, \ i{=}2, \\ 2i-3, \ i \text{ odd } 3 \le i \le n, \\ 2i-2, \ i \text{ even } 4 \le i \le n. \end{cases}$$

$$f(w_iv_i) = \begin{cases} 2i, \ i{=}1{,}2, \\ 2i-2, \ i \text{ odd } 3 \le i \le n, \\ 2i-3, \ i \text{ even } 4 \le i \le n. \end{cases}$$

$$f(w_iw_{i+1}) = \begin{cases} 2i+1, \ i{=}1{,}2, \\ 2i-1, \ 3 \le i \le n-1. \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 3, \ i{=}1, \\ 2i-2, \ 3 \le i \le n-1. \end{cases}$$

$$f(u_iw_{i+1}) = \begin{cases} 2, \ i{=}1, \\ 2i-1, \ 3 \le i \le 6, \\ 2i-1, \ i \text{ even } 6 \le i \le n-1, \\ 2i-2, \ i \text{ odd } 7 \le i \le n-1. \end{cases}$$

$$f(v_iw_{i+1}) = \begin{cases} 3, \ i{=}1, \\ 4, \ i{=}2, \\ 2i-2, \ 3 \le i \le 6, \\ 2i-2, \ i \text{ even } 6 \le i \le n-1, \\ 2i-1, \ i \text{ odd } 7 \le i \le n-1. \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} 2, \ i{=}2, \\ 2i-2, \ i \text{ even } 4 \le i \le n-1. \end{cases}$$

Based on the proof of the upper and lower bounds of the triangle ribbon ladder graph (LSP_n), the maximum value of point labels which are even integers and edge labels which are positive integers is $\lceil \frac{6n-4}{3} \rceil$, for $6n - 4 \not\equiv 2, 3 \pmod 6$, and $\lceil \frac{6n-4}{3} \rceil + 1$, for $6n - 4 \equiv 2 \pmod 6$.

Therefore, the edge weights are obtained as follows,

$$w_t(u_iw_i) = \begin{cases} 1, \ i\text{=}1, \\ 6i-4, \ i \text{ even } 2 \le i \le n, \\ 6i-5, \ i \text{ odd } 3 \le i \le n. \end{cases}$$

$$w_t(u_iu_{i+1}) = 6i, \ i \text{ even } 2 \le i \le n-1.$$

$$w_t(v_iw_{i+1}) = \begin{cases} 5, \ i\text{=}1, \\ 6i-2, \ 2 \le i \le 6, \\ 6i-1, \ i \text{ odd } 7 \le i \le n-1, \\ 6i-2, \ i \text{ even } 8 \le i \le n-1. \end{cases}$$

$$w_t(u_iw_{i+1}) = \begin{cases} 2, \ i\text{=}1, \\ 6i-1, \ 2 \le i \le 6, \\ 6i-2, \ i \text{ odd } 7 \le i \le n-1, \\ 6i-1, \ i \text{ even } 8 \le i \le n-1. \end{cases}$$

$$w_t(v_iv_{i+1}) = \begin{cases} 7, \ i\text{=}1, \\ 6i, \ i \text{ odd } 3 \le i \le n-1. \end{cases}$$

$$w_t(w_iw_{i+1}) = 6i-3, \ 1 \le i \le n-1.$$

$$w_t(w_iv_i) = \begin{cases} 2i+2, \ i\text{=}1,2, \\ 6i-4, \ i \text{ odd } 3 \le i \le n, \\ 6i-5, \ i \text{ even } 4 \le i \le n. \end{cases}$$

From the research that has been done, it appears that all the edge weights of the triangle ribbon ladder graph (LSP_n) are different, the lower and upper bounds are the same as $res(LSP_n)$. Thus f satisfies the element of edge irregular reflexive labeling and has the strength $res(LSP_n)$ according to Theorem 2. Thus the theorem is proven. □

4. Conclusions

Based on the research, we conclude that

(1) for all positive integers $r \ge 2$,

$$res(C(N_r)) = \begin{cases} \lceil \frac{9r}{3} \rceil, & \text{for } 9r \not\equiv 2,3 \pmod 6, \\ \lceil \frac{9r}{3} \rceil + 1, & \text{for } 9r \equiv 3 \pmod 6. \end{cases}$$

(2) for all positive integers $n \ge 2$,

$$res(LSP_n) = \begin{cases} \lceil \frac{6n-4}{3} \rceil, & \text{for } 6n-4 \not\equiv 2,3 \pmod 6, \\ \lceil \frac{6n-4}{3} \rceil + 1, & \text{for } 6n-4 \equiv 2 \pmod 6. \end{cases}$$

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