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GENERALIZED SOLUTIONS FOR TIME ψ-FRACTIONAL WAVE EQUATION

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Abstract. This paper focuses on the fractional wave problem with the use of a new fractional derivative in Colombeau algebra. Using Banach's fixed point theorem and Laplace transforms, we give and prove the integral solution of the problem. In Colombeau's algebra, the existence and uniqueness of the solution are demonstrated using the Gronwall lemma.

Keywords: Fractional wave equation, $\psi-$Caputo derivative, Generalized solution, Laplace transforms, Singular data, Colombeau algebra.

AMS Subject Classification: 35D99, 35K67, 34A45, 46F05.

1. Introduction

The fractional wave equation is a mathematical model that describes the propagation of waves in a medium with a fractional derivative. It is a generalization of the classical wave equation, which assumes that the medium is described by a second-order derivative. The fractional wave equation has applications in various fields, including acoustics, electromagnetics, and seismology. It can also be used to model the behavior of viscoelastic materials, such as polymers, which exhibit fractional-order behavior.

The solution to the fractional wave equation involves the use of fractional calculus, which is a branch of mathematics that deals with derivatives and integrals of non-integer order. The solution can be obtained using various techniques, including numerical methods and Fourier analysis.

Overall, the fractional wave equation provides a more accurate description of wave propagation in complex media and has important applications in many fields of science and engineering.

In recent years many researchers have centered on the study of phenomena whose modeling is given by nonlinear differential equations with a singularity, to do this, it is necessary

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to define the multiplication of two distributions in a manner that is consistent with the standard multiplication, this naturally led us to study the problem within Colombeaus algebra. This algebra which is commutative, associative, differential in which we can imbed the space of distributions so that the product of the infinitely differentiable functions and the regular derivative are satisfied [10].

The introduction of Caputo derivative into algebra of generalized functions was motivated by the opportunity of solving nonlinear issues with singularities and derivatives of any real order. In order to offer a sense of our situation, we use a special space of Colombeau algebra type which is a commutative, associative differential algebra where we're able to inject D' (space of distributions) so that the smooth function product as well as the normal distribution derivative have been preserved [20, 24].

In this paper we introduce a new method for solving the fractional wave problem with initial data are singular (singular distribution) as we can see in the following

$$\left\{ \begin{array}{l} D_{\psi}^{c} x(y,t) + c^2 \frac{d^2}{dy^2} x(y,t) - m x(y,t) = 0, \quad t \in [0,T] \\ x(y,0) = a_0(y) \end{array} \right. \tag{1}$$

Where a_0, c, m are singular generalized functions and D_{ψ}^{c} is $\psi-$Caputo derivative of order α, $\alpha \in]1;2]$.

The study is structured as follows: in section 2 we mention some concepts of Colombeau's algebra, in section 3 we will give and demonstrate the existence of $\psi-$Caputo derivative in Colombeau algebra $\mathcal{G}$, in section 4 we gave and demonstrated the integral solution of the issue, in section 5, we demonstrated the existence and uniqueness of the solution in Colombeau algebra.

2. Preliminaries

In this section we will introduce basic notations and definitions from Colombeau theory (see also [20]).

Definition 1. [20] *$\mathcal{A}_0(\mathbb{R}^n)$ is a set of functions ϕ in $C_0^{\infty}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(t)dt = 1$. For $q \in \mathbb{N}, \mathcal{A}_q(\mathbb{R}^n) = \left\{\phi \in \mathcal{A}_0 : \int_{\mathbb{R}^n} t^i \phi(t)dt = 0, 0 < |i| \leq q\right\}$, where $t^i = t_1^{i_1} \cdots t_n^{i_n}$.*

In [20] sets

$$\overline{\mathcal{A}}_q(\mathbb{R}^n) = \{\Phi(x_1, .., x_n) = \Phi(x_1)..\Phi(x_n) : \phi(x_i) \in \mathcal{A}_q(\mathbb{R})\},$$

are used because of applications to initial value problems. We shall follow the Colombeau original definition.

Obviously, if $\phi \in \mathcal{A}_q, q \in \mathbb{N}_0$, then for every $\varepsilon > 0, \phi_{\varepsilon}(x) = \frac{1}{\varepsilon^n}\phi\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^n$, belongs to $\mathcal{A}_q$. If $\phi \in \mathcal{A}_0$, then its support number $d(\phi)$ is defined by

$$d(\phi) = \sup\{|x| : \phi(x) \neq 0\}.$$

$\mathcal{E}(\Omega)$ represents the set of

$$R : \mathcal{A}_0 \times \Omega \to \mathbb{C}, (\Phi, x) \mapsto R(\Phi, x),$$

which are in $C^{\infty}(\Omega)$ for every fixed ϕ. In the other words elements of $\mathcal{E}$ are functions $R : \mathcal{A}_0 \to C^{\infty}$. Note that for any $f \in C^{\infty}$, the mapping

$$(\phi, x) \mapsto f(x), (\phi, x) \in \mathcal{A}_0 \times \Omega,$$

defines an element in $\mathcal{E}(\Omega)$ which does not depend on ϕ. Conversely, if an element F in $\mathcal{E}(\Omega)$ does not depend on $\Phi \in \mathcal{A}_0$, we have:

$$F(\Phi, x) = F(\Psi, x), \quad x \in \Omega, \text{ for every } \Phi, \Psi \in \mathcal{A}_0,$$

then it defines a function $f \in C^\infty(\Omega)$,

$$f(x) = F(\Phi, x), x \in \Omega, \text{ for every } \phi \in \mathcal{A}_0.$$

In this sense, we identify $C^\infty(\Omega)$ with the corresponding subspace of $\mathcal{E}(\Omega)$.

Definition 2. [25] *A component $R \in \mathcal{E}(\Omega)$ is moderate if $\forall L \subset\subset \Omega$, $\alpha \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that for every $\Phi \in \mathcal{A}_N$, $\exists \eta > 0$ and $C > 0$ such that:*

$$\|\partial^\alpha R(\Phi_\epsilon, x)\| \leq C\epsilon^{-N} \qquad x \in L, 0 < \epsilon < \eta.$$

The ensemble of all mild components is expressed as $\mathcal{E}_M(\Omega)$.

Definition 3. [25] *An element $R \in \mathcal{E}_0(\mathbb{C})$ is moderate if $\exists N \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$, $\exists \eta > 0$, $C > 0$ such that:*

$$\|R(\phi_\varepsilon)\| < C\varepsilon^{-N}, 0 < \varepsilon < \eta.$$

The ensemble of mild components is expressed by $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$).

Definition 4. [25] *A component $R \in \mathcal{E}_M(\Omega)$ is named null if for every $L \subset\subset \Omega$ and every $\alpha \in \mathbb{N}_0^n$, $\exists N \in \mathbb{N}_0$ and $\{a_q\} \in \Gamma$ such that for every $q \geq N$ and every $\phi \in \mathcal{A}_q$, $\exists \eta > 0$ and $C > 0$ such that:*

$$\|\partial^\alpha R(\phi_\varepsilon, x)\| \leq C\varepsilon^{a_q - N} \qquad x \in L, 0 < \varepsilon < \eta.$$

The ensemble of null components is expressed by $\mathcal{N}(\Omega)$.

Definition 5. [25] *The spaces of generalized functions $\mathcal{G}(\Omega)$ expressed by*

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$$

The following description describes what the term "association" means in Colombeau algebra.

Definition 6. [25] *Let $f, g \in \mathcal{G}(\mathbb{R})$.*
We said that f,g are associated if $\forall$ $h(\varphi_\epsilon, x)$ and $m(\varphi_\epsilon, x)$ and arbitrary $\xi \in \mathcal{D}(\mathbb{R})$ there is a $n \in \mathbb{N}$ such that $\forall \varphi(x) \in \mathcal{A}_n(\mathbb{R})$, we have:

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \|h(\varphi_\epsilon, x) - m(\varphi_\epsilon, x)\| \xi(x) dx = 0$$

and we denoted by $f \approx g$.

3. $\psi-$Fractional Derivative in colombeau algebra

Let $(f_\epsilon(t))_\epsilon$ be a representative of a Colombeau generalized function $f(t) \in \mathcal{G}(\mathbb{R}^+)$ and let $\psi \in \mathcal{C}^n(\mathbb{R}^+)$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in \mathbb{R}^+$.

The ψ-Caputo fractional derivative of $(f_\epsilon(t))_\epsilon$, is defined by

$$D^c_\psi f_\epsilon(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (\psi(t) - \psi(s))^{n-\alpha-1} f_\epsilon^{[n]}(s) \psi'(s) ds, & \alpha \in]n-1, n[, \\ f_\epsilon^{(n)}(t) = f_\epsilon^{[n]}(t) = (\frac{1}{\psi'(t)} \frac{d}{dt})^n f_\epsilon(t), & \alpha = n, \end{cases} \tag{2}$$

$n \in \mathbb{N}, \epsilon \in (0,1)$

Lemma 1. *Let $(f_\epsilon(t))_\epsilon$ be a representative of $f(t) \in \mathcal{G}(\mathbb{R}^+)$. Then, for every $\alpha > 0$, $\sup_{t \in [0,T]} \mid D^c_\psi f_\epsilon(t) \mid$ has a moderate bound.*

Proof. Fix $\epsilon \in (0,1)$.
Let $\alpha \in]n-1,n]$,

Then,
$\sup_{t\in[0,T]} \mid D^{c}_{\psi} f_{\epsilon}(t) \mid \leq \frac{1}{\Gamma(n-\alpha)} \sup_{t\in[0,T]} \int_0^t \mid (\psi(t)-\psi(s))^{n-\alpha-1} f_{\epsilon}^{[n]}(s)\psi'(s) \mid ds$

$$= \frac{1}{\Gamma(n-\alpha)} \sup_{s\in[0,T]} \mid f_{\epsilon}^{[n]}(s) \mid \sup_{t\in[0,T]} \mid \frac{(\psi(t)-\psi(0))^{n-\alpha}}{n-\alpha} \mid$$

$$\leq \frac{1}{\Gamma(n-\alpha)} \frac{T^{n-\alpha}}{n-\alpha} \sup_{s\in[0,T]} \mid f_{\epsilon}^{[n]}(s) \mid .$$

Since $f(t) \in \mathcal{G}([0,+\infty))$, as a result $\sup_{s\in[0,T]} \mid f_{\epsilon}^{[n]}(s) \mid$ has a moderate bound.

Thus, $\exists M \in \mathbb{N}$, such that

$$\sup_{t\in[0,T]} \mid D^{c}_{\psi} f_{\epsilon}(t) \mid = \mathcal{O}\left(\varepsilon^{-M}\right), \quad \varepsilon \to 0$$

Then, $\sup_{t\in[0,T]} \mid D^{c}_{\psi} f_{\epsilon}(t) \mid$ has a moderate bound, $\forall \alpha > 0$.

□

Lemma 2. *Let $(f_{1\epsilon}(t))_{\epsilon}$, $(f_{2\epsilon}(t))_{\epsilon}$ be two distinct representatives of $f(t) \in \mathcal{G}(\mathbb{R}^{+})$. Then, for every $\alpha > 0$, $\sup_{t\in[0,T]} \mid D^{c}_{\psi} f_{1\epsilon}(t) - D^{c}_{\psi} f_{2\epsilon}(t) \mid$ is negligible.*

Proof. Fix $\epsilon \in (0,1)$.
Let $\alpha \in]n-1,n]$,
Then,

$$\sup_{t\in[0,T]} \mid D^{c}_{\psi} f_{1\epsilon}(t) - D^{c}_{\psi} f_{2\epsilon}(t) \mid \leq \frac{1}{\Gamma(n-\alpha)} \frac{T^{n-\alpha}}{n-\alpha} \sup_{s\in[0,T]} \mid f_{1\epsilon}^{[n]}(s) - f_{2\epsilon}^{[n]}(s) \mid .$$

Since $(f_{1\epsilon}(t))_{\epsilon}$ and $(f_{2\epsilon}(t))_{\epsilon}$ represent the same Colombeau generalized function $f(t)$, so $\sup_{s\in[0,T]} \mid f_{1\epsilon}^{[n]}(s) - f_{2\epsilon}^{[n]}(s) \mid$ is negligible, then for all p$\in \mathbb{N}$

$$\sup_{t\in[0,T]} \mid D^{c}_{\psi} f_{1\epsilon}(t) - D^{c}_{\psi} f_{2\epsilon}(t) \mid = \mathcal{O}\left(\varepsilon^{-p}\right), \quad \varepsilon \to 0$$

Therefore, $\sup_{t\in[0,T]} \mid D^{c}_{\psi} f_{1\epsilon}(t) - D^{c}_{\psi} f_{2\epsilon}(t) \mid$ is negligible. □

We may now initiate the ψ-Caputo fractional derivative of a Colombeau generalized function on $\mathbb{R}^{+}$ after establishing the first two lemmas.

Definition 7. *Let $f(t) \in \mathcal{G}(\mathbb{R}^{+})$ be a Colombeau function on $\mathbb{R}^{+}$.
The $\psi-$Caputo fractional derivative of $f(t)$, utilizing the notation $D^{c}_{\psi} f(t) = \left[\left(D^{c}_{\psi} f_{\epsilon}(t)\right)_{\epsilon}\right]$, $\alpha > 0$, is the element of $\mathcal{G}(\mathbb{R}^{+})$ satisfying (2).*

Remark 1. *For $\alpha \in]n-1,n]$.
The first derivative of $(\mathrm{d}/\mathrm{d}t) D^{c}_{\psi} f_{\epsilon}(t)$ is $(\mathrm{d}/\mathrm{d}t) D^{c}_{\psi} f_{\epsilon}(t) =$
$(1/\Gamma(1-\alpha)) \left[\int_0^t \left(\frac{\psi\prime(s)}{(\psi(t)-\psi(s))^{\alpha+1-n}} f_{\varepsilon}^{[n+1]}(s)\right) \mathrm{d}s + \frac{\psi\prime(0)}{(\psi(t)-\psi(0))^{\alpha+1-n}} f_{\epsilon}^{[m]}(0)\right]$ and it is not defined in zero, unless $f_{\epsilon}^{[m]}(0) = 0$.*

Theorem 1. *Let $f(t) \in \mathcal{G}$ be a Colombeau function. The $\psi-$Caputo fractional derivative $D^{\alpha}_{\psi} f(t)$ is a Colombeau generalized function, if $f_{\varepsilon}^{[n]}(0) = f_{\epsilon}^{[n+1]}(0) = f_{\epsilon}^{[n+2]}\ (0) = \cdots = 0$.*

Proof. Let $\alpha \in]n-1,n]$.
In Lemma 1, we proved that $\sup_{t\in[0,T]} \mid D^{\alpha}_{\psi} f_{\epsilon}(t) \mid$ has a moderate limit for indefinite Colombeau generalized function. To get a moderate limit for the initial derivative (d/d $t)D^{\alpha}_{\psi} f_{\epsilon}(t)$ we utilize the expression acquired in Remark 1 and for $f_{\epsilon}^{[n]}(0) = 0$, we obtain

$$(\mathrm{d}/\mathrm{d}t) D^{\alpha}_{\psi} f_{\epsilon}(t) = (1/\Gamma(1-\alpha)) \int_0^t \left(\frac{\psi\prime(S)}{(\psi(t)-\psi(s))^{\alpha+1-n}} f_{\varepsilon}^{[n+1]}(s) \right) \mathrm{d}s$$

.
Now, in the same way as in Lemma 1 we acquires a moderate limit for $\sup_{t\in[0,T]} \mid (\mathrm{d}/\mathrm{d}t) D^{\alpha}_{\psi} f_{\epsilon}(t) \mid$.
Using the conditions, higher-order derivatives can be estimated similarly. $f_{\epsilon}^{[n]}(0) = f_{\epsilon}^{[n+1]}(0) = f_{\epsilon}^{[n+2]}(0) = \cdots = 0$.
Finally, if $f_{\epsilon}^{[n]}(0) = 0$, therefore, it follows that for each $\alpha > 0$, all derivatives of $D^{\alpha}_{\psi} f_{\epsilon}(t)$ have moderate representations. □

Definition 8. *Let $(f_{\epsilon}(t))_{\epsilon}$ be a representative of $f(t) \in \mathcal{G}(\mathbb{R}^+)$.
The regularized $\psi-$Caputo fractional derivative of $(f_{\epsilon}(t))_{\epsilon}$, is given by*

$$\tilde{D}^{c}_{\psi} f_{\epsilon}(t) = \begin{cases} \left(D^{c}_{\psi} f_{\epsilon}(t) * \varphi_{\varepsilon} \right)(t), & \alpha \in]n-1,n] \\ f_{\epsilon}^{(n)}(t) = f_{\epsilon}^{[n]}(t) = (\frac{1}{\psi'(t)} \frac{\mathrm{d}}{dt})^n f_{\epsilon}(t), & \alpha = n, \end{cases} \quad (3)$$

$n \in \mathbb{N}, \epsilon \in (0,1)$.

where $D^{c}_{\psi} f_{\epsilon}(t)$ is provided by (2).

The convolution in (3) is $\left(D^{c}_{\psi} f_{\epsilon}(t) * \varphi_{\epsilon} \right)(t) = \int_0^{\infty} D^{c}_{\psi} f_{\epsilon}(t) \varphi_{\epsilon}(t-s) \mathrm{d}s$.

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Definition 9. *Define the Mittag-Leffler function by:*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k\alpha+\beta)}.$$

Definition 10. *Describe the Laplace transform of a function g by*

$$\mathcal{L}(g(x))(s) = \int_0^{+\infty} e^{-sx} g(x) dx.$$

Proposition 1. *Let f and g two functions, we have*

$$\mathcal{L}\left((f*g)(x)\right)(s) = \mathcal{L}\left(f(x)\right)(s) \mathcal{L}\left(g(x)\right)(s).$$

Definition 11. (1) *The Gamma function is given by*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \forall x > 0$$

(2) *The $\mathbb{B}$ function is described by*

$$\forall x, y > 0, \quad \mathbb{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

Proposition 2. (1) $\forall x, y \in \mathbb{R}^*_+ \times \mathbb{R}^*_+$, $\mathbb{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.
(2) *For all $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.*

Definition 12. *The Wright type function is represented by*

$$\begin{aligned}\phi_\alpha(x) &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma(-\alpha n+1-\alpha)} \\ &= \sum_{n=0}^{\infty} \frac{(-x)^n \Gamma(\alpha(n+1))\sin(\pi(n+1)\alpha)}{n!}\end{aligned}$$

for $\alpha \in (0,1)$ *and* $x \in \mathbb{C}$.

Proposition 3. *The Wright function* ϕ_α *is a complete function with the following characteristics:*

(i) $\int_0^\infty \phi_\alpha(\theta)\theta^r d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ *for* $r > -1$*;*
(ii) $\phi_\alpha(\theta) \geq 0$ *for* $\theta \geq 0$ *and* $\int_0^\infty \phi_\alpha(\theta) d\theta = 1$
(iii) $\int_0^\infty \phi_\alpha(\theta)e^{-z\theta} d\theta = E_\alpha(-z), \quad z \in \mathbb{C}$*;*
(iv) $\alpha \int_0^\infty \theta\phi_\alpha(\theta)e^{-z\theta} d\theta = E_{\alpha,\alpha}(-z), \quad z \in \mathbb{C}$.

Definition 13. *We proceed with the observed one-sided steady probability density in*

$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\theta^{-\alpha k-1}\frac{\Gamma(\alpha k+1)}{k!}\sin(k\pi\alpha), \quad \theta \in (0,\infty)$$

And we have,

$$\int_0^\infty e^{-\lambda\theta}\rho_\alpha(\theta)d\theta = e^{-\lambda^\alpha}, \text{ where } \alpha \in (0,1). \tag{4}$$

Lemma 3. *Let* $f : \mathcal{C}(J,X) \to \mathcal{C}(J,X)$ *be continuous.*

The issue (1) is equal to the mild equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(s))^{1-\alpha}}\psi'(s)Ax(s)ds, \quad t \in J, \tag{5}$$

With:
$x : D(A) \longrightarrow D(A)$ *offered that the integral in 5 exists, and* $A = c^2\frac{d^2}{dy^2} - m$.

We will need the following lemma.

Lemma 4. *For all* $\alpha \in]n-1, n] \quad n \in \mathbb{N}$ *and* $s > 0$*, and let* $\phi \in \mathcal{C}^n(\mathbb{R}^+)$ *be an increasing function with* $\phi'(t) \neq 0$ *for all* $t \in \mathbb{R}^+$*. We have,*
1) $s^{\alpha-1}(s^\alpha - A)^{-1} = \mathcal{L}\big(\int_0^\infty \rho_\alpha(\theta)T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right)d\theta\big)(s)$,

2) $(s^\alpha - A)^{-1}X(s) = \mathcal{L}\big(\Big(\int_0^\tau\int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\phi(\tau)-\phi(s))^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\phi(\tau)-\phi(s))^\alpha}{\theta^\alpha}\right)$
$x(s)\phi'(s)d\theta ds\big)\big)(s)$.

With,
$X(s) = \int_0^\infty e^{-\lambda(\phi(s)-\phi(0))}x(s)\phi'(s)ds$ *and* $A = c^2\frac{d^2}{dy^2} - m$.

Proof. 1) For $s > 0$,
$s^{\alpha-1}(s^\alpha - A)^{-1} = s^{\alpha-1}\int_0^\infty e^{-s^\alpha}T(\tau)d\tau = \alpha\int_0^\infty (s\hat{t})^{\alpha-1}e^{-(s\hat{t})^\alpha}T\left(\hat{t}^\alpha\right)dt$
$Where \quad \{T\}_{t\geq 0} \quad is \quad C_0 - semigroup \quad defined \quad by$
$Ax = \lim_{t\longrightarrow 0^+}\frac{T(t)x-x}{x}$ *and* $(\lambda^\alpha I - A)^{-1}x = \int_0^\infty exp(-\lambda^\alpha t)T(t)xdt$

$Putting \quad \hat{t} = \phi(t) - \phi(0), \quad we \quad have$

$= \alpha \int_0^\infty s^{\alpha-1}(\phi(t)-\phi(0))^{\alpha-1} e^{-(s(\phi(t)-\phi(0))^{\alpha}} \times T\left((\phi(t)-\phi(0))^{\alpha}\right)\psi'(t)dt$

$= \int_0^\infty -\frac{1}{s}\frac{d}{dt}\left(e^{-(s(\phi(t)-\phi(0)))^{\alpha}}\right) T\left((\phi(t)-\phi(0))^{\alpha}\right)dt.$

Using (4), we get

$= \int_0^\infty \int_0^\infty \theta \rho_\alpha(\theta) e^{-s(\phi(t)-\phi(0))^{\theta}} T\left((\phi(t)-\phi(0))^{\alpha}\right)\psi'(t)d\theta dt$

$= \int_0^\infty e^{-s(\phi(t)-\phi(0))}\left(\int_0^\infty \rho_\alpha(\theta) T\left(\frac{(\phi(t)-\phi(0))^{\alpha}}{\theta^{\alpha}}\right)d\theta\right)\psi'(t)dt$

$= \mathcal{L}\big(\int_0^\infty \rho_\alpha(\theta) T\left(\frac{(\phi(t)-\phi(0))^{\alpha}}{\theta^{\alpha}}\right)d\theta\big)(s)$
2) For $s > 0$,

$$
\begin{aligned}
\left(s^{\alpha}-A\right)^{-1} X(s) &= \int_0^\infty e^{-s^{\alpha}\tau} T(\tau) X(s) d\tau \\
&= \alpha \int_0^\infty \hat{\tau}^{\alpha-1} e^{-(s\tau))^{\alpha}} T\left(\hat{\tau}^{\alpha}\right) X(s) d\tau \\
& Where \quad \{T\}_{t\geq 0} \quad is \quad C_0-semigroup \quad defined \quad by \\
& Ax = \lim_{t\longrightarrow 0^+} \frac{T(t)x-x}{x} \quad and \\
& (\lambda^{\alpha} I - A)^{-1} x = \int_0^\infty exp(-\lambda^{\alpha} t) T(t) x dt \\
& Putting \quad \hat{t} = \phi(t)-\phi(0), \quad we \quad have \\
& = \int_0^\infty \alpha(\phi(\tau)-\phi(0))^{\alpha-1} e^{-(s(\phi(\tau)-\phi(0)))^{\alpha}} \\
& \times T\left((\phi(\tau)-\phi(0))^{\alpha}\right)\phi'(\tau) X(s) d\tau \\
&= \int_0^\infty \int_0^\infty \alpha(\phi(\tau)-\phi(0))^{\alpha-1} e^{-(s(\phi(\tau)-\phi(0)))^{\alpha}} \\
& T\left((\phi(\tau)-\phi(0))^{\alpha}\right) \times e^{-(\lambda(\phi(r)-\phi(0)))} x(r)\psi'(r)\phi'(\tau) dr d\tau,
\end{aligned}
$$

Using (4), we get

$= \int_0^\infty \int_0^\infty \int_0^\infty \alpha(\phi(\tau)-\phi(0))^{\alpha-1}\rho_\alpha(\theta) e^{-s(\phi(\tau)-\phi(0))\theta'} T\left((\phi(\tau)-\phi(0))^{\alpha}\right)$
$\times\, e^{-s(\phi(r)-\phi(0))} x(r)\phi'(r)\phi'(\tau) d\theta dr d\tau$

$= \int_0^\infty \int_0^\infty \int_0^\infty \alpha e^{-s(\phi(\tau)+\phi(r)-2\phi(0))}\frac{(\phi(\tau)-\phi(0))^{\alpha-1}}{\theta^{\alpha}}\rho_\alpha(\theta)$
$\times\, T\left(\frac{(\phi(\tau)-\phi(0))^{\alpha}}{\theta^{\alpha}}\right) x(r)\phi'(r)\phi'(\tau) d\theta dr d\tau$

$= \int_0^\infty \int_t^\infty \int_0^\infty \alpha e^{-s(\phi(\tau)-\phi(0))}\rho_\alpha(\theta)\frac{(\phi(t)-\phi(0))^{\alpha-1}}{\theta^{\alpha}} T\left(\frac{(\phi(t)-\phi(0))^{\alpha}}{\theta^{\alpha}}\right)$
$x\left(\phi^{-1}(\phi(\tau)-\phi(t)+\phi(0))\right)$
$\phi'(\tau)\phi'(t) d\theta d\tau dt$

$= \int_0^\infty \int_0^\tau \int_0^\infty \alpha e^{-s(\phi(\tau)-\phi(0))}\rho_\alpha(\theta)\frac{(\phi(t)-\phi(0))^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right)$
$x\left(\phi^{-1}(\phi(\tau)-\phi(t)+\phi(0))\right)\phi'(\tau)$
$\phi'(t)d\theta dt d\tau$

$= \int_0^\infty e^{-s(\phi(\tau)-\phi(0))}\left(\int_0^\tau \int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\phi(\tau)-\phi(r))^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\phi(\tau)-\phi(r))^\alpha}{\theta^\alpha}\right)\right.$
$x(r)\phi'(r)d\theta dr) \times \phi'(\tau)d\tau.$

$= \mathcal{L}\big(\left(\int_0^\tau \int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\phi(\tau)-\phi(r))^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\phi(\tau)-\phi(r))^\alpha}{\theta^\alpha}\right)\right.$
$x(r)\phi'(r)d\theta dr)\big)(s)$

□

Proposition 4. *If*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)Ax(s)ds,$$

holds, then we have

$$x(t) = E(t)x_0 + \alpha\int_0^t E(t)(\psi(t)-\psi(s))^{\alpha-1}x(s)\psi'(s)ds.$$

With,
$E(t) = \int_0^\infty \phi_a(\theta)T\left((\psi(t)-\psi(0))^\alpha\theta\right)d\theta$

Proof. Since $x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(s))^{1-\alpha}}\psi'(s)Ax(s)ds$, using the Laplace transform, we obtain

$$\begin{aligned}\mathcal{L}\big(x(t)\big)(s) &= \mathcal{L}\big(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(\tau))^{1-\alpha}}\psi'(\tau)Ax(\tau)d\tau\big)(s)\\ &= \mathcal{L}(x_0)(s) + \mathcal{L}\big(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(\tau))^{1-\alpha}}\psi'(\tau)Ax(\tau)d\tau\big)(s)\\ &= \frac{x_0}{s} + \mathcal{L}\big(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(\tau))^{1-\alpha}}\psi'(\tau)Ax(\tau)d\tau\big)(s)\\ &= \frac{x_0}{s} + \frac{1}{s^\alpha}A\mathcal{L}\left(x(t)\right)(s)\\ &= \frac{x_0}{s} + \frac{1}{s^\alpha}A\int_0^\infty e^{-\lambda(\psi(s)-\psi(0))}x(s)\psi'(s)ds\end{aligned}$$

We can deduce
$\mathcal{L}\big(x(t)\big)(s) = s^{\alpha-1}\left(s^\alpha - A\right)^{-1}x_0 + \left(s^\alpha - A\right)^{-1}X(s).$
Now, use the lemma 4, then
$\mathcal{L}\big(x(t)\big)(s) = \mathcal{L}\big(\int_0^\infty \rho_\alpha(\theta)T\left(\frac{(\psi(t)-\psi(0))^\alpha}{\theta^\alpha}\right)d\theta\big)(s)x_0 +$
$\mathcal{L}\big(\left(\int_0^\tau \int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\psi(\tau)-\psi(s))^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\psi(\tau)-\psi(s))^\alpha}{\theta^\alpha}\right)\right.$
$x(s)\psi'(s)d\theta ds)\big)(s)$

We can now invert the Laplace transform to obtain the result

$\forall x \in X$, characterize operators $S_{\psi}^{\alpha}(t,s)$ and $T_{\psi}^{\alpha}(t,s)$ by

$$S_{\psi}^{\alpha}(t,s)x = \int_{0}^{\infty} \phi_{\alpha}(\theta) T\left((\psi(t)-\psi(s))^{\alpha}\theta\right) x d\theta$$

And

$$T_{\psi}^{\alpha}(t,s)x = \alpha \int_{0}^{\infty} \theta \phi_{a}(\theta) T\left((\psi(t)-\psi(s))^{\alpha}\theta\right) x d\theta$$

for $0 \leq s \leq t \leq T$. □

Lemma 5. *S_{ψ}^{α} and T_{ψ}^{α} provide the following characteristics :*

(i) The operators $S_{\psi}^{\alpha}(t,s)$ and $T_{\psi}^{\alpha}(t,s)$ are strongly continuous for all $t \geq s \geq 0$, that is, for every $x \in X$ and $0 \leq s \leq t_1 < t_2 \leq T$ we have

$$| S_{\psi}^{\alpha}(t_2,s)x - S_{\psi}^{\alpha}(t_1,s)x\| \to 0 \text{ and } | T_{\psi}^{\alpha}(t_2,s)x - T_{\psi}^{\alpha}(t_1,s)x\| \to 0$$

as $t_1 \to t_2$.

(ii) For any fixed $t \geq s \geq 0, S_{\psi}^{\alpha}(t,s)$ and $T_{\psi}^{\alpha}(t,s)$ are bounded linear operators with

$$| S_{\psi}^{\alpha}(t,s)(x)\| \leq M\|x\| \text{ and } \left\|T_{\dot{\psi}}^{\alpha}(t,s)(x)\right\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\| = \frac{M}{\Gamma(\alpha)}\|x\|$$

for all $x \in X$.

Proof. Similar demonstration existe in [**?**] □

5. Existence and Uniqueness of the Solution in colombeau algebra

In this section consider the following fractional wave problem:

$$\begin{cases} D_{\psi}^{c}x(y,t) + Ax(y,t) = 0, & t \in [0,T] \\ x(y,0) = a_0(y) \end{cases}$$

with $a_0(y) \in D'(\mathbb{R}^n)$ *and* $A = c^2 \frac{d^2}{dy^2} - m.$

Now we will transform the problem in the Colombeau algebra (see section 2).

$$\begin{cases} D_{\psi}^{c}x_{\varepsilon}(y,t) + A_{\varepsilon}x_{\varepsilon}(y,t) = 0 \quad y \in \mathbb{R}^n, \quad t \geq 0 \\ \quad x_{\varepsilon}(y,0) = a_{0,\varepsilon}(y) \end{cases} \tag{6}$$

with $a_{0,\varepsilon}(y)$ is the regularizetion of $a_0(y)$, and $A = [(A_{\varepsilon})] = [(c_{\epsilon}^2 \frac{d^2}{dy^2} - m_{\epsilon}]$ is the infinitesimal generator of $\{T_{\epsilon}(t)\}$ $C_0 - semigroup$.

Theorem 2. *If the generalized operators S_{ψ}^{α} and T_{ψ}^{α} verify the Lemma (5). Then the problem (6) has a unique solution in $\mathcal{G}(\mathbb{R}^n \times \mathbb{R}^+)$.*

Proof. **Existence**

The integral solution of the problem (6) is Given through the previous section:

$$\begin{aligned} x_\varepsilon(t) &= \int_0^\infty \phi_{\varepsilon,a}(\theta) T\left((\psi_\varepsilon(t)-\psi_\varepsilon(0))^\alpha\theta\right) x_{\varepsilon,0} d\theta \\ &+ \alpha \int_0^t \int_0^\infty \phi_{\varepsilon,a}(\theta)(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1} T\left((\psi_\varepsilon(t)-\psi_\varepsilon(0))^\alpha\theta\right) \\ & x_\epsilon(s)\psi_\varepsilon'(s) d\theta ds. \\ &= S_\psi^\alpha(t,s) x_{\varepsilon,0} + \int_0^t (\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1} T_\alpha^\psi(t,s) x_\epsilon(s)\psi_\varepsilon' ds \end{aligned}$$

Which implies that:
$\|x_\varepsilon(t,.)\| \leq \left\|S_{\psi,\varepsilon}^\alpha(t,0)x_{\varepsilon,0}\right\| + \int_0^t \left\|(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1} T_{\psi,\varepsilon}^\alpha(t,s)x_\epsilon(s)\psi_\varepsilon'(s)\right\| ds$
$\leq M\|x_{\varepsilon,0}\| + \int_0^t (\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1} \left\|T_{\psi,\varepsilon}^\alpha(t,s)x_\epsilon(s)\right\| \psi_\varepsilon'(s) ds$
Then

$$\|x_\varepsilon(t,.)\| \leq M\|x_{\varepsilon,0}\| + \frac{M}{\Gamma(\alpha)} \int_0^t (\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1} \|x_\varepsilon(s,.)\| \psi_\varepsilon'(s) ds$$

By the Granwall's inequality

$$\|x_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq M\|a_{\varepsilon,0}\| \times \exp\left(\frac{M}{\Gamma(\alpha+1)}(\psi_\varepsilon(t)-\psi_\varepsilon(0))^\alpha\right).$$

Since $\psi_\varepsilon \in G(\mathbb{R}^+)$, $a_{0,\epsilon} \in \mathcal{G}(\mathbb{R}^n)$ there exist $K \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \|x_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}\left(\varepsilon^{-K}\right), \quad \varepsilon \to 0$$

So

$$x \in \mathcal{G}(\mathbb{R}^+ \times \mathbb{R}^n)$$

Uniqueness

Let's say there are two solutions $x_{1,\epsilon}(t,.)$, $x_{2,\epsilon}(t,.)$ to the problem (6), consequently :

$$\begin{cases} D_\psi^c x_{1,\epsilon}(y,t) + A_\epsilon x_{1,\epsilon}(y,t) - D_\psi^c x_{2,\epsilon}(y,t) - A_\epsilon x_{2,\epsilon}(y,t) = 0 \\ y \in \mathbb{R}^n, \quad t \geq 0 \\ x_{1,\epsilon}(y,0) - x_{2,\epsilon}(y,0) = N_{0,\epsilon}(y) \end{cases}$$

Then:

$$\begin{cases} D_\psi^c \left(x_{1,\epsilon}(y,t) - x_{2,\epsilon}(y,t)\right) + A_\epsilon \left(x_{1,\epsilon}(y,t) - x_{2,\epsilon}(y,t)\right) = 0 \\ y \in \mathbb{R}^n, \quad t \geq 0 \\ x_{1,\epsilon}(y,0) - x_{2,\epsilon}(y,0) = N_{0,\epsilon}(y) \end{cases} \tag{7}$$

With $(N_{0,\varepsilon})_\varepsilon \in \mathcal{N}(\mathbb{R}^+)$.
The integral solution of the equation (7) is:
$x_\varepsilon(t) = \int_0^\infty \phi_{\varepsilon,a}(\theta) T\left((\psi_\varepsilon(t)-\psi_\varepsilon(0))^\alpha\theta\right) N_{0,\epsilon}(y) d\theta$
$+\alpha \int_0^t \int_0^\infty \phi_{\varepsilon,a}(\theta)(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1} T\left((\psi_\varepsilon(t)-\psi_\varepsilon(0))^\alpha\theta\right)$
$\times \left(x_{1,\varepsilon}(s) - x_{2,\varepsilon}(s)\right) \psi_\varepsilon'(s) d\theta ds.$
$= S_\psi^\alpha(t,s) N_{0,\epsilon}(y) + \int_0^t (\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1} T_\psi^\alpha(t,s) \times \left(x_{1,\varepsilon}(s) - x_{2,\varepsilon}(s)\right) \psi_\varepsilon' ds.$

Then

$$\begin{aligned}
&\|x_{1,\varepsilon}(t,.)-x_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \le \left\|S^\alpha_{\psi,\varepsilon}(t,0)N_{0,\varepsilon}(.)\right\| \\
&+\textstyle\int_0^t \left\|(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}T^\alpha_{\psi,\varepsilon}(t,s)\left(x_{1,\varepsilon}(s)-x_{2,\varepsilon}(s)\right)\psi'_\varepsilon(s)\right\| ds \\
&\le M\left\|N_{0,\varepsilon}(.)\right\| + \textstyle\int_0^t (\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\left\|T^\alpha_{\psi,\varepsilon}(t,s)\left(x_{1,\varepsilon}(s)-x_{2,\varepsilon}(s)\right)\right\| \\
&\times\psi'_\varepsilon(s)ds \\
&\le M\left\|N_{0,\varepsilon}(.)\right\| + \tfrac{M}{\Gamma(\alpha)}\textstyle\int_0^t (\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\left\|(x_{1,\varepsilon}(s)-x_{2,\varepsilon}(s))\right\| \times \psi'_\varepsilon(s)ds
\end{aligned}$$

So

$$\begin{aligned}
&\|x_{1,\varepsilon}(t,.)-x_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \le M\left\|N_{0,\varepsilon}(.)\right\| + \tfrac{M}{\Gamma(\alpha)}\textstyle\int_0^t (\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1} \\
&\times\left\|(x_{1,\epsilon}(s,.)-x_{2,\epsilon}(s,.))\right\|\psi'_\varepsilon(s)ds
\end{aligned}$$

Using the Granwall's inequalit
$\|x_{1,\varepsilon}(t,.)-x_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \le M\left\|N_{0,\varepsilon}(.)\right\| \times \exp\left(\frac{M}{\Gamma(\alpha+1)}(\psi_\varepsilon(T)-\psi_\varepsilon(0))^\alpha\right)$
Since
$\psi_\varepsilon \in G(\mathbb{R}^+), \quad (N_{0,\varepsilon})_\varepsilon \in \mathcal{N}(\mathbb{R}^+)$, then for every $q\in\mathbb{N}$ such that:

$$\sup_{t\in[0,T]} \|x_{1,\epsilon}(t,.)-x_{2,\epsilon}(t,.)\|_{L^\infty} = \mathcal{O}\left(\varepsilon^q\right) \quad \varepsilon\to 0$$

So,

$$x_{1,\epsilon} \approx x_{2,\epsilon}.$$

□

6. Conclusion

In this work, we have investigated the ψ-Caputo fractional wave equation within the framework of Colombeau generalized function algebras, offering a rigorous approach to handle singular initial data and distributional coefficients. By constructing a new form of fractional derivative adapted to generalized functions, we successfully established the existence and uniqueness of solutions through fixed-point arguments and fractional semigroup theory. Furthermore, we derived the integral representation of the solution and analyzed its properties using Laplace transforms and the Mittag-Leffler function.

This study contributes to the broader understanding of fractional wave phenomena in complex media where traditional distribution theory fails due to the presence of singularities. The generalized setting allows for a consistent multiplication of distributions and provides a natural setting to model real-world systems exhibiting memory and hereditary effects.

Future research could extend this approach to nonlinear problems, multidimensional domains, or systems involving coupled fractional differential equations. Moreover, numerical simulations and applied case studies would help to better understand the physical implications and validate the theoretical results presented here.

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