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HYBRID BI-IDEALS AND HYBRID QUASI-IDEALS IN ORDERED SEMIRINGS

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Abstract. Fuzzy set theory has been proven to be a powerful tool for dealing with uncertainty in decision-making processes. This theory addresses uncertain parameters. Soft set theory is another mathematical concept used for managing uncertainty in decision-making processes and imprecision. Integrating the ideas of a fuzzy set and a soft set, Jun et al. established the concept of hybrid structure. We should emphasize that hybrid structures combine soft and fuzzy set theories. The main objective of this paper is to explore the concept of hybrid bi-ideals and hybrid quasi-ideals in ordered semirings. In addition, we construct an example of hybrid bi-ideal and hybrid quasi-ideal in an ordered semiring. We provide various properties of hybrid bi-ideal and hybrid quasi-ideal in ordered semirings.

Keywords: Ordered semiring, regular, hybrid structure, hybrid bi-ideal, and hybrid quasi-ideal.

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1. Introduction

H. S. Vandiver [20] explored the concept of semiring in 1934. Semirings have proven successful in dealing with problems in several fields of applied mathematics and information sciences. Semirings play an important role in both geometry and pure mathematics. Semirings applications included optimization theory, graph theory, theory of discrete event dynamical systems, generalized fuzzy computation, automata theory, formal language theory, coding theory, etc. Ideals play an important role in advanced research and algebraic structural applications. Numerous mathematicians used the concept of ideals and demonstrated significant results and characterizations of algebraic systems.

An ordered semiring is a generalization of semirings. In 2011, Gan and Jiang [7] addressed the idea of an ordered semiring as a semiring with a partially ordered relation on

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the semiring such that the relation is compatible with the semiring operations. Numerous researchers have developed notions relating to the maximum ordered, and minimum ideals of an ordered semiring. Satyt Patchakhieo and Bundit Pibalijommee [18] investigated the left and right ordered ideals of ordered semirings and examined their various properties.

In 1965, Zadeh [21] presented the notion of fuzzy sets, which are helpful for describing imprecise or unclear data. Fuzzy theory is gaining popularity because it can solve numerous problems that cannot be solved by classical mathematics, and thus has received significant attention. Numerous fields, including mathematics, computer science, artificial intelligence, medical sciences, economics, statistics, and neural networks, have recently found extensive applications of fuzzy set theory. Many methods appear for extending fuzzy sets, like bipolar fuzzy sets, hesitant fuzzy sets, vague sets, intuitionistic fuzzy sets, etc. Several authors studied fuzzy logic in various algebraic systems (see [9, 10, 11, 19]).

In 1999, Molodtsov [15] demonstrated a whole new method of modelling vagueness and uncertainty in mathematics. This is known as soft set theory, which is free from the difficulties affecting existing methods. Applications of soft set theory can be found across a wide range of disciplines, such as probability theory, measurement theory, Riemann integration, Perron integration, operations research, game theory, and the smoothness of functions.

Maji et al. [12] presented the idea of fuzzy soft sets as a fusion of fuzzy and soft sets, in 2001. It provides an effective tool for measuring uncertainty as well as a significant representation of vagueness in natural language. Therefore, the idea of a fuzzy soft set is extremely beneficial in modelling problems involving more parameters with uncertainty and ambiguity. The study of soft set theory is currently progressing quickly.

In a set of parameters over an initial universe set, Jun et al. [8] presented the notion of hybrid structures and investigated numerous types of properties. Using this method, they developed the concepts of a hybrid field, a hybrid subalgebra, and a hybrid linear space. Anis et al. [1] explored the idea of hybrid ideals and hybrid subsemigroups in semigroups. Elavarasan et al. [2] defined the regularity of semigroups in terms of hybrid generalized bi-ideals and examined the characteristics of a hybrid generalized bi-ideal, an extension of a hybrid bi-ideal. Several semigroup properties were obtained using hybrid ideals in semigroups (see [4, 5, 13, 16, 17]).

In addition, hybrid ideals, hybrid $k-$ closures [3], hybrid $k-$ ideals [6], and hybrid subsemimodules [14] in semirings were discussed and several results were obtained. In this work, we utilize hybrid structures in ordered semirings to introduce hybrid bi-ideals and hybrid quasi-ideals. Additionally, we construct an example that illustrates the concept of hybrid bi-ideals and hybrid quasi-ideals within an ordered semiring. Moreover, we prove that every hybrid right (respectively, left) ideal in an ordered semiring is a hybrid quasi-ideal of an ordered semiring.

2. Preliminaries

In this section, we discuss the fundamental definitions related to ordered semiring theory and hybrid structure theory that are required for this article.

Definition 2.1. [7] *Let $\mathscr{X}(\neq \emptyset)$, "$+$", "$\cdot$" two binary operations defined on $\mathscr{X}$. Then $\mathscr{X}$ is known as semiring if*

(i) $(\mathscr{X}, +)$ is a semigroup,

(ii) $(\mathscr{X}, \cdot)$ is a semigroup,

(iii) $(x_0 + f_0) \cdot a_0 = x_0 \cdot a_0 + f_0 \cdot a_0$ and $x_0 \cdot (f_0 + a_0) = x_0 \cdot f_0 + x_0 \cdot a_0 \; \forall \; x_0, f_0, a_0 \in \mathscr{X}$.

Definition 2.2. [7] *An ordered semiring is a semiring $\mathscr{X}$ equipped with a partial order $\leq$ such that the operation is monotonic and constant 0 is the least element of $\mathscr{X}$.*

Throughout this paper, $\mathfrak{P}(Q)$ denotes the power set of a set Q and $(\mathscr{X}, +, \cdot, \leq)$ denotes the ordered semiring.

Definition 2.3. [7] *Let $\mathcal{Z} \in \mathfrak{P}(\mathscr{X})$. Then $\mathcal{Z}$ is defined as a left (respectively, right) ideal of $\mathscr{X}$ if it fulfils:*
(i) $y_1 + y_2 \in \mathcal{Z}$,
(ii) $fy_1 \in \mathcal{Z}$ (respectively, $y_1 f \in \mathcal{Z}$) for all $y_1, y_2 \in \mathcal{Z}$ and $f \in \mathscr{X}$.
$\mathcal{Z}$ is known as an ideal of $\mathscr{X}$ if it is both a left and a right ideal of $\mathscr{X}$.

Definition 2.4. [7] *A subset $U(\neq \emptyset) \in \mathscr{X}$ is known as*
(i) a bi-ideal of $\mathscr{X}$ if U is closed under addition and $U\mathscr{X}U \subseteq U$.
(ii) a quasi-ideal of $\mathscr{X}$ if U is closed under addition and $\mathscr{X}U \cap U\mathscr{X} \subseteq U$.

Definition 2.5. [7] *Let $\mathscr{X}$ be an ordered semiring and $\emptyset \neq \mathcal{Z} \subseteq \mathscr{X}$. Then $\mathcal{Z}$ is defined as a left (respectively, right) ordered ideal if it fulfils:*
(i) $\mathcal{Z}$ is left (respectively, right) ideal of $\mathscr{X}$.
(ii) For $w \in \mathscr{X}$, if $w \leq l$ for some $l \in \mathcal{Z}$, then $w \in \mathcal{Z}$.
If $\mathcal{Z}$ is both a left and right ordered ideal of $\mathcal{Z}$, then it is termed as an ordered ideal of $\mathscr{X}$.

Definition 2.6. [10] *An ordered semiring $\mathscr{X}$ is said to be regular if for each $w \in \mathscr{X}$, $\exists\ h \in \mathscr{X}$ such that $w \leq whw$.*

Definition 2.7. [10] *An ordered semiring $\mathscr{X}$ is said to be intra-regular if for each $w \in \mathscr{X}$ $\exists\ h, n \in \mathscr{X}$ such that $w \leq hw^2n$.*

Definition 2.8. [8] *For an universal set $\mathscr{A}$, a hybrid structure in $\mathscr{X}$ over $\mathscr{A}$ is $\tilde{d}_{\beth} := (\tilde{d}, \beth) : \mathscr{X} \to \mathfrak{P}(\mathscr{A}) \times [0,1],\ t \mapsto (\tilde{d}(t), \beth(t))$, where $\tilde{d} : \mathscr{X} \to \mathfrak{P}(\mathscr{A})$ and $\beth : \mathscr{X} \to [0,1]$ are mappings.*

The collection of all hybrid structures in $\mathscr{X}$ over $\mathscr{A}$ is represented by $\mathcal{H}(\mathscr{X})$. Clearly $(\mathcal{H}(\mathscr{X}), \ll)$ is a poset, where the relation $\ll$ defined on $\mathcal{H}(\mathscr{X})$ is described as below:

$$\Big(\forall\ \tilde{d}_{\beth}, \tilde{z}_{\varrho} \in \mathcal{H}(\mathscr{X})\Big), \Big(\tilde{d}_{\beth} \ll \tilde{z}_{\varrho} \Leftrightarrow \tilde{d} \mathbin{\tilde{\subseteq}} \tilde{z}, \beth \succeq \varrho\Big),$$

where $\tilde{d} \mathbin{\tilde{\subseteq}} \tilde{z}$ means that $\tilde{d}(q) \subseteq \tilde{z}(q)$ and $\beth \succeq \varrho$ means that $\beth(q) \geq \varrho(q)\ \forall\ q \in \mathscr{X}$.

Definition 2.9. [1] *Let $\tilde{n}_{\beth}, \tilde{z}_{\daleth} \in \mathcal{H}(\mathscr{X})$. Then*
(i) the hybrid union $\tilde{n}_{\beth} \Cup \tilde{z}_{\daleth}$ is described as $\tilde{n}_{\beth} \Cup \tilde{z}_{\daleth} : \mathscr{X} \to \mathfrak{P}(\mathscr{A}) \times [0,1], s \mapsto ((\tilde{n} \tilde{\cup} \tilde{z})(s), (\beth \wedge \daleth)(s))$, where

$$\tilde{n} \tilde{\cup} \tilde{z} : \mathscr{X} \to \mathfrak{P}(\mathscr{A}), s \mapsto \tilde{n}(s) \cup \tilde{z}(s),$$
$$\beth \wedge \daleth : \mathscr{X} \to [0,1], s \mapsto \beth(s) \wedge \daleth(s)\ for\ s \in \mathscr{X}.$$

(ii) the hybrid intersection $\tilde{n}_{\beth} \Cap \tilde{z}_{\daleth}$ is described as $\tilde{n}_{\beth} \Cap \tilde{z}_{\daleth} : \mathscr{X} \to \mathfrak{P}(\mathscr{A}) \times [0,1], s \mapsto ((\tilde{n} \tilde{\cap} \tilde{z})(s), (\beth \vee \daleth)(s))$, where

$$\tilde{n} \tilde{\cap} \tilde{z} : \mathscr{X} \to \mathfrak{P}(\mathscr{A}), s \mapsto \tilde{n}(s) \cap \tilde{z}(s),$$
$$\beth \vee \daleth : \mathscr{X} \to [0,1], s \mapsto \beth(s) \vee \daleth(s)\ for\ s \in \mathscr{X}.$$

(iii) the hybrid composition $\tilde{n}_{\beth} \odot \tilde{z}_{\daleth}$ *is described as* $\tilde{n}_{\beth} \odot \tilde{z}_{\daleth} := (\tilde{n} \circ \tilde{z}, \beth \circ \daleth)$*, where*

$$(\tilde{n} \circ \tilde{z})(q) = \begin{cases} \bigcup\limits_{q \leq ly} \{\tilde{n}(l) \cap \tilde{z}(y)\} & if\ q \leq ly \\ \emptyset & otherwise; \end{cases}$$

$$(\beth \circ \daleth)(q) = \begin{cases} \bigwedge\limits_{q \leq ly} \{\beth(l) \vee \daleth(y)\} & if\ q \leq ly \\ 1 & otherwise \end{cases}$$

for $q, l, y \in \mathscr{X}$.

Definition 2.10. *For* $\tilde{c}_{\varsigma} \in \mathcal{H}(\mathscr{X})$ *and* $D \in \mathfrak{P}(\mathscr{X}) \backslash \{\emptyset\}$, $\chi_D(\tilde{c}_{\varsigma})$ *is the characteristic hybrid structure in* $\mathscr{X}$ *over* $\mathscr{A}$*, defined as follows:* $\chi_D(\tilde{c}_{\varsigma}) = (\chi_D(\tilde{c}), \chi_D(\varsigma)) : \mathscr{X} \longrightarrow \mathfrak{P}(\mathscr{A}) \times [0,1],\ z \mapsto (\chi_D(\tilde{c})(z), \chi_D(\varsigma)(z))$*, where*

$$\chi_D(\tilde{c}) : \mathscr{X} \to \mathfrak{P}(\mathscr{A}), z \mapsto \begin{cases} \mathscr{A} & if\ z \in D \\ \emptyset & otherwise; \end{cases}$$

$$\chi_D(\varsigma) : \mathscr{X} \to [0,1], z \mapsto \begin{cases} 0 & if\ z \in D \\ 1 & otherwise \end{cases}$$

for any $z \in \mathscr{X}$.

3. Properties of hybrid bi-ideals and hybrid quasi-ideals

In this section, we define the notions of hybrid bi-ideals and hybrid quasi-ideals in ordered semirings and explore their various properties.

Definition 3.1. *Let* $\tilde{y}_{\daleth} \in \mathcal{H}(\mathscr{X})$*. Then* $\tilde{y}_{\daleth}$ *is defined as a hybrid left (respectively, right) ideal of* $\mathscr{X}$ *if*

(i) $\begin{pmatrix} \tilde{y}(q_1 + d_1) \supseteq \tilde{y}(q_1) \cap \tilde{y}(d_1) \\ \daleth(q_1 + d_1) \leq \daleth(q_1) \vee \daleth(d_1) \end{pmatrix}$,

(ii) $\begin{pmatrix} \tilde{y}(q_1 d_1) \supseteq \tilde{y}(d_1) (\textit{respectively, } \tilde{y}(q_1 d_1) \supseteq \tilde{y}(q_1)) \\ \daleth(q_1 d_1) \leq \daleth(d_1) (\textit{respectively, } \daleth(q_1 d_1) \leq \daleth(q_1)) \end{pmatrix}$,

(iii) $q_1 \leq d_1 \implies \begin{pmatrix} \tilde{y}(q_1) \supseteq \tilde{y}(d_1) \\ \daleth(q_1) \leq \daleth(d_1) \end{pmatrix}$ *for any* $q_1, d_1 \in \mathscr{X}$.

If $\tilde{y}_{\daleth}$ *is both a hybrid left and a hybrid right ideal of* $\mathcal{Z}$*, then it is termed as a hybrid ideal of* $\mathscr{X}$.

Definition 3.2. *Let* $\tilde{b}_{\gimel} \in \mathcal{H}(\mathscr{X})$*. Then* $\tilde{b}_{\gimel}$ *is defined as a hybrid bi-ideal of* $\mathscr{X}$ *if*

(i) $\begin{pmatrix} \tilde{b}(q_1 + d_1) \supseteq \tilde{b}(q_1) \cap \tilde{b}(d_1) \\ \gimel(q_1 + d_1) \leq \gimel(q_1) \vee \gimel(d_1) \end{pmatrix}$,

(ii) $\begin{pmatrix} \tilde{b}(q_1 d_1) \supseteq \tilde{b}(q_1) \cap \tilde{b}(d_1) \\ \gimel(q_1 d_1) \leq \gimel(q_1) \vee \gimel(d_1) \end{pmatrix}$,

(iii) $\begin{pmatrix} \tilde{b}(q_1 d_1 k_1) \supseteq \tilde{b}(q_1) \cap \tilde{b}(k_1) \\ \gimel(q_1 d_1 k_1) \leq \gimel(q_1) \vee \gimel(k_1) \end{pmatrix}$,

(iv) $q_1 \leq d_1 \implies \begin{pmatrix} \tilde{b}(q_1) \supseteq \tilde{b}(d_1) \\ \gimel(q_1) \leq \gimel(d_1) \end{pmatrix}$ *for any* $q_1, d_1, k_1 \in \mathscr{X}$.

Definition 3.3. *Let* $\tilde{w}_{\daleth} \in \mathcal{H}(\mathscr{X})$*. Then* $\tilde{w}_{\daleth}$ *is described as hybrid quasi-ideal of* $\mathscr{X}$ *if*

(i) $\begin{pmatrix} \tilde{w}(f_1 + n_1) \supseteq \tilde{w}(f_1) \cap \tilde{w}(n_1) \\ \daleth(f_1 + n_1) \leq \daleth(f_1) \vee \daleth(n_1) \end{pmatrix}$,

(ii) $(\tilde{w}_\urcorner \odot \chi_{\mathscr{X}}) \Cap (\chi_{\mathscr{X}} \odot \tilde{w}_\urcorner) \ll \tilde{w}_\urcorner$,

(iii) $f_1 \leq n_1 \implies \begin{pmatrix} \tilde{w}(f_1) \supseteq \tilde{w}(n_1) \\ \urcorner(f_1) \leq \urcorner(n_1) \end{pmatrix}$ *for any* $f_1, n_1 \in \mathscr{X}$.

Below are examples of the hybrid bi-ideal and hybrid quasi-ideals in an ordered semiring.

Example 3.1. *Let* $\mathscr{X} = \{0, m, t, g\}$ *with the ordered relation* $0 < g < t < m$. *Define the binary operations* "$+$" *and* "$\cdot$" *on* $\mathscr{X}$ *as follows:*

$+$	0	m	t	g
0	0	m	t	g
m	m	m	t	g
t	t	t	t	g
g	g	g	g	g

$\cdot$	0	m	t	g
0	0	0	0	0
m	0	m	m	m
t	0	t	t	t
g	0	g	g	g

Then $(\mathscr{X}, +, \cdot, <)$ *forms an ordered semiring. For any* $L, A, W, B \in \mathfrak{P}(\mathscr{A}) \setminus \{\emptyset\}$ *and* $l, a, w, b \in [0, 1)$, *define* $\tilde{r}_\daleth \in \mathcal{H}(\mathscr{X})$ *by* $\tilde{r}(0) = L, \tilde{r}(m) = A, \tilde{r}(t) = W$ *and* $\tilde{r}(g) = B$; $\daleth(0) = l, \daleth(m) = a, \daleth(t) = w$ *and* $\daleth(g) = b$ *with* $A \subset W \subset B \subset L$ *and* $a > w > b > l$. *Then* $\tilde{r}_\daleth$ *is a hybrid bi-ideal of* $\mathscr{X}$.

Example 3.2. *Let* $\mathscr{X} = \{h, b, m, p\}$ *be a set with the ordered relation* $h < b < m < p$ *Define the binary operations* "$+$" *and* "$\cdot$" *on* $\mathscr{X}$ *as follows:*

$+$	h	b	m	p
h	h	b	m	p
b	b	b	m	p
m	m	m	m	p
p	p	p	p	p

$\cdot$	h	b	m	p
h	h	h	h	h
b	h	b	b	b
m	h	b	b	b
p	h	b	b	b

Then $(\mathscr{X}, +, \cdot, <)$ *forms an ordered semiring. For* $T, Z, E, S \in \mathfrak{P}(\mathscr{A}) \setminus \{\emptyset\}$ *and* $t, z, e, s \in [0, 1)$, *define a hybrid structure* $\tilde{k}_\varphi$ *of* $\mathscr{X}$ *by* $\tilde{k}(h) = T, \tilde{k}(b) = Z, \tilde{k}(m) = S$ *and* $\tilde{k}(p) = E$; $\varphi(h) = t, \varphi(b) = z, \varphi(m) = s$ *and* $\varphi(p) = e$ *with* $E \subset S \subset Z \subset T$ *and* $e > s > z > t$. *Then* $\tilde{k}_\varphi$ *is a hybrid quasi-ideal of* $\mathscr{X}$.

Theorem 3.1. *Hybrid intersection of the collection of hybrid bi-ideals of* $\mathscr{X}$ *is also a hybrid bi-ideal of* $\mathscr{X}$.

Proof. Assume that $\{(\tilde{u}_\daleth)_{z_1} \mid z_1 \in I\}$ is the collection of hybrid bi-ideals of $\mathscr{X}$ and $q, b, d \in \mathscr{X}$. Then

$$\bigcap_{z_1 \in I} \tilde{u}_{z_1}(q+b) \supseteq \bigcap_{z_1 \in I} \{\tilde{u}_{z_1}(q) \cap \tilde{u}_{z_1}(b)\} = \left(\bigcap_{z_1 \in I} \tilde{u}_{z_1}(q)\right) \cap \left(\bigcap_{z_1 \in I} \tilde{u}_{z_1}(b)\right);$$

$$\bigvee_{z_1 \in I} \daleth_{z_1}(q+b) \leq \bigvee_{z_1 \in I} \{\daleth_{z_1}(q) \vee \daleth_{z_1}(b)\} = \left(\bigvee_{z_1 \in I} \daleth_{z_1}(q)\right) \vee \left(\bigvee_{z_1 \in I} \daleth_{z_1}(b)\right).$$

$$\bigcap_{z_1 \in I} \tilde{u}_{z_1}(qb) \supseteq \bigcap_{z_1 \in I} \{\tilde{u}_{z_1}(q) \cap \tilde{u}_{z_1}(b)\} = \left(\bigcap_{z_1 \in I} \tilde{u}_{z_1}(q)\right) \cap \left(\bigcap_{z_1 \in I} \tilde{u}_{z_1}(b)\right);$$

$$\bigvee_{z_1 \in I} \daleth_{z_1}(qb) \leq \bigvee_{z_1 \in I} \{\daleth_{z_1}(q) \vee \daleth_{z_1}(b)\} = \left(\bigvee_{z_1 \in I} \daleth_{z_1}(q)\right) \vee \left(\bigvee_{z_1 \in I} \daleth_{z_1}(b)\right).$$

Also,

$$\bigcap_{z_1 \in I} \tilde{u}_{z_1}(qbd) \supseteq \bigcap_{z_1 \in I} \{\tilde{u}_{z_1}(q) \cap \tilde{u}_{z_1}(d)\} = \left(\bigcap_{z_1 \in I} \tilde{u}_{z_1}(q)\right) \cap \left(\bigcap_{z_1 \in I} \tilde{u}_{z_1}(d)\right);$$

$$\bigvee_{z_1 \in I} \gimel_{z_1}(qbd) \leq \bigvee_{z_1 \in I} \{\gimel_{z_1}(q) \vee \gimel_{z_1}(d)\} = \left(\bigvee_{z_1 \in I} \gimel_{z_1}(q)\right) \vee \left(\bigvee_{z_1 \in I} \gimel_{z_1}(d)\right).$$

Suppose $q \leq b$. Then $\tilde{u}_{z_1}(q) \supseteq \tilde{u}_{z_1}(b)$ and $\gimel_{z_1}(q) \leq \gimel_{z_1}(b)$ for all $z_1 \in I$ which imply $\bigcap_{z_1 \in I} \tilde{u}_{z_1}(q) \supseteq \bigcap_{z_1 \in I} \tilde{u}_{z_1}(b)$ and $\bigvee_{z_1 \in I} \gimel_{z_1}(q) \leq \bigvee_{z_1 \in I} \gimel_{z_1}(b)$. Hence $\Cap_{z_1 \in I} (\tilde{u}_\gimel)_{z_1}$ is a hybrid bi-ideal of $\mathscr{X}$. $\square$

Theorem 3.2. *Let $\{(\tilde{b}_\varsigma)_{v_0} \mid v_0 \in I\}$ be the collection of bi-ideals of $\mathscr{X}$. Then $\Cup_{v_0 \in I} (\tilde{b}_\varsigma)_{v_0}$ is a hybrid bi-ideal of $\mathscr{X}$.*

Proof. Let $\{(\tilde{b}_\varsigma)_{v_0} \mid v_0 \in I\}$ be the collection of bi-ideals of $\mathscr{X}$ and $m, z, l \in \mathscr{X}$. Then we have

$$\bigcup_{v_0 \in I} \tilde{b}_{v_0}(m+z) \supseteq \bigcup_{v_0 \in I} \{\tilde{b}_{v_0}(m) \cap \tilde{b}_{v_0}(z)\} = \left(\bigcup_{v_0 \in I} \tilde{b}_{v_0}(m)\right) \cap \left(\bigcup_{v_0 \in I} \tilde{b}_{v_0}(z)\right);$$

$$\bigwedge_{v_0 \in I} \varsigma_{v_0}(m+z) \leq \bigwedge_{v_0 \in I} \{\varsigma_{v_0}(m) \vee \varsigma_{v_0}(z)\} = \left(\bigwedge_{v_0 \in I} \varsigma_{v_0}(m)\right) \vee \left(\bigwedge_{v_0 \in I} \varsigma_{v_0}(z)\right).$$

Also,

$$\bigcup_{v_0 \in I} \tilde{b}_{v_0}(mz) \supseteq \bigcup_{v_0 \in I} \{\tilde{b}_{v_0}(m) \cap \tilde{b}_{v_0}(z)\} = \left(\bigcup_{v_0 \in I} \tilde{b}_{v_0}(m)\right) \cap \left(\bigcup_{v_0 \in I} \tilde{b}_{v_0}(z)\right);$$

$$\bigwedge_{v_0 \in I} \varsigma_{v_0}(mz) \leq \bigwedge_{v_0 \in I} \{\varsigma_{v_0}(m) \vee \varsigma_{v_0}(z)\} = \left(\bigwedge_{v_0 \in I} \varsigma_{v_0}(m)\right) \vee \left(\bigwedge_{v_0 \in I} \varsigma_{v_0}(z)\right).$$

And

$$\bigcup_{v_0 \in I} \tilde{b}_{v_0}(mzl) \supseteq \bigcup_{v_0 \in I} \{\tilde{b}_{v_0}(m) \cap \tilde{b}_{v_0}(l)\} = \left(\bigcup_{v_0 \in I} \tilde{b}_{v_0}(m)\right) \cap \left(\bigcup_{v_0 \in I} \tilde{b}_{v_0}(l)\right);$$

$$\bigwedge_{v_0 \in I} \varsigma_{v_0}(mzl) \leq \bigwedge_{v_0 \in I} \{\varsigma_{v_0}(m) \vee \varsigma_{v_0}(l)\} = \left(\bigwedge_{v_0 \in I} \varsigma_{v_0}(m)\right) \vee \left(\bigwedge_{v_0 \in I} \varsigma_{v_0}(l)\right).$$

Suppose $m \leq z$. Then $\tilde{b}_{v_0}(m) \supseteq \tilde{b}_{v_0}(z)$ and $\varsigma_{v_0}(m) \leq \varsigma_{v_0}(z)$ for all $v_0 \in I$ which imply $\bigcup_{v_0 \in I} \tilde{b}_{v_0}(m) \supseteq \bigcup_{v_0 \in I} \tilde{b}_{v_0}(z)$ and $\bigwedge_{v_0 \in I} \varsigma_{v_0}(m) \leq \bigwedge_{v_0 \in I} \varsigma_{v_0}(z)$. Hence $\Cup_{v_0 \in I} (\tilde{b}_\varsigma)_{v_0}$ is a hybrid bi-ideal of $\mathscr{X}$. $\square$

Proposition 3.1. *Every hybrid right (respectively, left) ideal of $\mathscr{X}$ is a hybrid quasi-ideal of $\mathscr{X}$.*

Proof. Let $\tilde{n}_{\varphi}$ be a hybrid right ideal of $\mathscr{X}$. Then, for $q, s, c \in \mathscr{X}$, $\tilde{n}(q+s) \supseteq \tilde{n}(q) \cap \tilde{n}(s); \varphi(q+s) \leq \varphi(q) \vee \varphi(s)$ and $\tilde{n}(qs) \supseteq \tilde{n}(q); \varphi(qs) \leq \varphi(q)$.

We now prove that $(\chi_{\mathscr{X}} \odot \tilde{n}_{\varphi}) \Cap (\tilde{n}_{\varphi} \odot \chi_{\mathscr{X}}) \ll \tilde{n}_{\varphi}$.

If $q \nleq sc$ for any $s, c \in \mathscr{X}$, then it is clear that $(\chi_{\mathscr{X}} \odot \tilde{n}_{\varphi})(q) \Cap (\tilde{n}_{\varphi} \odot \chi_{\mathscr{X}})(q) \ll \tilde{n}_{\varphi}(q)$.

Assume that $q \leq sc$ for some $s, c \in \mathscr{X}$. Then $\tilde{n}(q) \supseteq \tilde{n}(sc) \supseteq \tilde{n}(s) = \tilde{n}(s) \cap \chi_{\mathscr{X}}(c)$; $\varphi(q) \leq \varphi(sc) \leq \varphi(s) = \varphi(s) \vee \chi_{\mathscr{X}}(c)$ which implies that $\tilde{n}(q) \supseteq \bigcup_{q \leq sc} \{\tilde{n}(s) \cap \chi_{\mathscr{X}}(c)\} = (\tilde{n} \circ \chi_{\mathscr{X}})(q)$; $\varphi(q) \leq \bigwedge_{q \leq sc} \{\varphi(s) \vee \chi_{\mathscr{X}}(c)\} = (\varphi \circ \chi_{\mathscr{X}})(q)$. So, $\tilde{n}(q) \supseteq (\tilde{n} \circ \chi_{\mathscr{X}})(q) \supseteq (\tilde{n} \circ \chi_{\mathscr{X}})(q) \cap (\chi_{\mathscr{X}} \circ \tilde{n})(q) = ((\chi_{\mathscr{X}} \circ \tilde{n}) \cap (\tilde{n} \circ \chi_{\mathscr{X}}))(q)$; $\varphi(q) \leq (\varphi \circ \chi_{\mathscr{X}})(q) \leq (\varphi \circ \chi_{\mathscr{X}})(q) \vee (\chi_{\mathscr{X}} \circ \varphi)(q) = ((\chi_{\mathscr{X}} \circ \varphi) \vee (\varphi \circ \chi_{\mathscr{X}}))(q)$.

Thus $(\chi_{\mathscr{X}} \odot \tilde{n}_{\varphi}) \Cap (\tilde{n}_{\varphi} \odot \chi_{\mathscr{X}}) \ll \tilde{n}_{\varphi}$ and hence $\tilde{n}_{\varphi}$ is a hybrid quasi-ideal of $\mathscr{X}$. □

Theorem 3.3. *Every hybrid quasi-ideal of $\mathscr{X}$ is a hybrid bi-ideal of $\mathscr{X}$.*

Proof. Let $\tilde{t}_{\daleth}$ be a hybrid quasi-ideal of $\mathscr{X}$. It is sufficient to prove that

$$\tilde{t}(qhm) \supseteq \tilde{t}(q) \cap \tilde{t}(m); \daleth(qhm) \leq \daleth(q) \vee \daleth(m) \ \forall\ q, h, m, s, d \in \mathscr{X}.$$

Since $\tilde{t}_{\daleth}$ is a hybrid quasi-ideal of $\mathscr{X}$, we get $\tilde{t}(qhm) \supseteq ((\chi_{\mathscr{X}} \circ \tilde{t}) \cap (\tilde{t} \circ \chi_{\mathscr{X}}))(qhm) = (\tilde{t} \circ \chi_{\mathscr{X}})(qhm) \cap (\chi_{\mathscr{X}} \circ \tilde{t})(qhm)$; $\daleth(qhm) \leq ((\chi_{\mathscr{X}} \circ \daleth) \vee (\daleth \circ \chi_{\mathscr{X}}))(qhm) = (\daleth \circ \chi_{\mathscr{X}})(qhm) \vee (\chi_{\mathscr{X}} \circ \daleth)(qhm)$. Now,

$$(\tilde{t} \circ \chi_{\mathscr{X}})(qhm) = \bigcup_{qhm \leq sd} \{\tilde{t}(s) \cap \chi_{\mathscr{X}}(d)\} = \tilde{t}(q) \cap \chi_{\mathscr{X}}(hm) = \tilde{t}(q);$$

$$(\daleth \circ \chi_{\mathscr{X}})(qhm) = \bigwedge_{qhm \leq sd} \{\daleth(s) \vee \chi_{\mathscr{X}}(d)\} = \daleth(q) \vee \chi_{\mathscr{X}}(hm) = \daleth(q).$$

Also,

$$(\chi_{\mathscr{X}} \circ \tilde{t})(qhm) = \bigcup_{qhm \leq sd} \{\chi_{\mathscr{X}}(s) \cap \tilde{t}(d)\} = \chi_{\mathscr{X}}(qh) \cap \tilde{t}(m) = \tilde{t}(m);$$

$$(\chi_{\mathscr{X}} \circ \daleth)(qhm) = \bigwedge_{qhm \leq sd} \{\chi_{\mathscr{X}}(s) \vee \daleth(d)\} = \chi_{\mathscr{X}}(qh) \vee \daleth(m) = \daleth(m).$$

Thus $\tilde{t}(qhm) \supseteq ((\chi_{\mathscr{X}} \circ \tilde{t}) \cap (\tilde{t} \circ \chi_{\mathscr{X}}))(qhm) = \tilde{t}(q) \cap \tilde{t}(m)$; $\daleth(qhm) \leq ((\chi_{\mathscr{X}} \circ \daleth) \vee (\daleth \circ \chi_{\mathscr{X}}))(qhm) = \daleth(q) \vee \daleth(m)$, hence $\tilde{t}_{\daleth}$ is a hybrid bi-ideal of $\mathscr{X}$. □

Combining Proposition 3.1 and Theorem 3.3, we have the following theorem.

Theorem 3.4. *Every hybrid right(respectively, left) ideal of $\mathscr{X}$ is a hybrid bi-ideal of $\mathscr{X}$.*

Proposition 3.2. *Let $\tilde{r}_{\urcorner}$ and $\tilde{m}_{\vartheta}$ be a hybrid right and a hybrid left ideals of $\mathscr{X}$ respectively. Then $\tilde{r}_{\urcorner} \Cap \tilde{m}_{\vartheta}$ is a hybrid quasi-ideal of $\mathscr{X}$.*

Proof. Let $\tilde{r}_{\urcorner}$ and $\tilde{m}_{\vartheta}$ be a hybrid right and a hybrid left ideals of $\mathscr{X}$ respectively. Then for any $b, l, n, g \in \mathscr{X}$, we get

$$(\tilde{r} \cap \tilde{m})(b+l) \supseteq ((\tilde{r} \cap \tilde{m})(b)) \cap ((\tilde{r} \cap \tilde{m})(l)); \ (\urcorner \vee \vartheta)(b+l) \leq ((\urcorner \vee \vartheta)(b)) \vee ((\urcorner \vee \vartheta)(l)).$$

Also

$$(\tilde{r} \circ \chi_{\mathscr{X}})(b) = \bigcup_{b \leq ng} \{\tilde{r}(n) \cap \chi_{\mathscr{X}}(g)\} = \tilde{r}(n) \subseteq \tilde{r}(ng) \subseteq \tilde{r}(b);$$

$$(\urcorner \circ \chi_{\mathscr{X}})(b) = \bigwedge_{b \leq ng} \{\urcorner(n) \vee \chi_{\mathscr{X}}(g)\} = \urcorner(n) \geq \urcorner(ng) \geq \urcorner(b).$$

And,

$$(\chi_{\mathscr{X}} \circ \tilde{m})(b) = \bigcup_{b \leq ng} \{\chi_{\mathscr{X}}(n) \cap \tilde{m}(g)\} = \tilde{m}(g) \subseteq \tilde{m}(ng) \subseteq \tilde{m}(b);$$

$$(\chi_{\mathscr{X}} \circ \vartheta)(b) = \bigwedge_{b \leq ng} \{\chi_{\mathscr{X}}(n) \vee \vartheta(g)\} = \vartheta(g) \geq \vartheta(ng) \geq \vartheta(b).$$

So $((\tilde{r}_\urcorner \Cap \tilde{m}_\vartheta) \odot \chi_{\mathscr{X}}) \Cap (\chi_{\mathscr{X}} \odot (\tilde{r}_\urcorner \Cap \tilde{m}_\vartheta)) \ll (\tilde{r}_\urcorner \odot \chi_{\mathscr{X}}) \Cap (\chi_{\mathscr{X}} \odot \tilde{m}_\vartheta) \ll \tilde{r}_\urcorner \Cap \tilde{m}_\vartheta$.

If $b \leq l$, then $\tilde{r}(b) \supseteq \tilde{r}(l)$; $\urcorner(b) \leq \urcorner(l)$ and $\tilde{m}(b) \supseteq \tilde{m}(l)$; $\vartheta(b) \leq \vartheta(l)$ which imply $(\tilde{r} \cap \tilde{m})(b) = \tilde{r}(b) \cap \tilde{m}(b) \supseteq \tilde{r}(l) \cap \tilde{m}(l)$; $(\urcorner \vee \vartheta)(b) = \urcorner(b) \vee \vartheta(b) \leq \urcorner(l) \vee \vartheta(l)$.

Hence $\tilde{r}_\urcorner \Cap \tilde{m}_\vartheta$ is a hybrid quasi-ideal of $\mathscr{X}$. □

Theorem 3.5. *Let $\tilde{s}_\varphi \in \mathcal{H}(\mathscr{X})$. Then $\tilde{s}_\varphi$ is a hybrid left (respectively, right) ideal of $\mathscr{X}$ if and only if for all $l, q \in \mathscr{X}$, we have*

(i) $\begin{pmatrix} \tilde{s}(l+q) \supseteq \tilde{s}(l) \cap \tilde{s}(q) \\ \varphi(l+q) \leq \varphi(l) \vee \varphi(q) \end{pmatrix}$,

(ii) $\chi_{\mathscr{X}} \odot \tilde{s}_\varphi \ll \tilde{s}_\varphi$ *(respectively,* $\tilde{s}_\varphi \odot \chi_{\mathscr{X}} \ll \tilde{s}_\varphi$*).*

Proof. Assume that $\tilde{s}_\varphi$ is a hybrid left ideal of $\mathscr{X}$. Then it is sufficient to show that the condition (ii) is satisfied. Let $l \in \mathscr{X}$. If $(\chi_{\mathscr{X}} \circ \tilde{s})(l) = \emptyset$; $(\chi_{\mathscr{X}} \circ \varphi)(l) = 1$, it is clear that $(\chi_{\mathscr{X}} \odot \tilde{s}_\varphi)(l) \ll \tilde{s}_\varphi(l)$. Otherwise, there exist elements $q, z \in \mathscr{X}$ such that $l \leq qz$. Then we have

$$(\chi_{\mathscr{X}} \circ \tilde{s})(l) = \bigcup_{l \leq qz} \{\chi_{\mathscr{X}}(q) \cap \tilde{s}(z)\} \subseteq \bigcup_{l \leq qz} \tilde{s}(z) \subseteq \tilde{s}(qz) \subseteq \tilde{s}(l);$$

$$(\chi_{\mathscr{X}} \circ \varphi)(l) = \bigwedge_{l \leq qz} \{\chi_{\mathscr{X}}(q) \vee \varphi(z)\} \geq \bigwedge_{l \leq qz} \varphi(z) \geq \varphi(qz) \geq \varphi(l).$$

This implies that $\chi_{\mathscr{X}} \odot \tilde{s}_\varphi \ll \tilde{s}_\varphi$.

Conversely, assume that the conditions (i) and (ii) are hold. Then it is sufficient to show that $\tilde{s}(lq) \supseteq \tilde{s}(q)$; $\varphi(lq) \leq \varphi(q)$ for any $l, q \in \mathscr{X}$. Now

$$\tilde{s}(lq) \supseteq (\chi_{\mathscr{X}} \circ \tilde{s})(lq) = \bigcup_{lq \leq tm} \{\chi_{\mathscr{X}}(t) \cap \tilde{s}(m)\} \supseteq \chi_{\mathscr{X}}(l) \cap \tilde{s}(q) = \tilde{s}(q);$$

$$\varphi(lq) \leq (\chi_{\mathscr{X}} \circ \varphi)(lq) = \bigwedge_{lq \leq tm} \{\chi_{\mathscr{X}}(t) \vee \varphi(m)\} \leq \chi_{\mathscr{X}}(l) \vee \varphi(q) = \varphi(q).$$

Hence $\tilde{s}_\varphi$ is a hybrid left ideal of $\mathscr{X}$. □

Lemma 3.1. [10] *Let $\mathscr{X}$ be an ordered semiring. Then the following conditions are equivalent:*

(i) $\mathscr{X}$ is regular,

(ii) $V = V\mathscr{X}V$ for every bi-ideal V of $\mathscr{X}$,

(iii) $T = T\mathscr{X}T$ for every quasi-ideal T of $\mathscr{X}$.

Theorem 3.6. *Let $\mathscr{X}$ be an ordered semiring. Then the following conditions are equivalent:*

(i) $\mathscr{X}$ is regular,

(ii) $\tilde{w}_\varsigma \ll \tilde{w}_\varsigma \odot \chi_{\mathscr{X}} \odot \tilde{w}_\varsigma$ for every hybrid bi-ideal $\tilde{w}_\varsigma$ of $\mathscr{X}$,

(iii) $\tilde{w}_\varsigma \ll \tilde{w}_\varsigma \odot \chi_{\mathscr{X}} \odot \tilde{w}_\varsigma$ for every hybrid quasi-ideal $\tilde{w}_\varsigma$ of $\mathscr{X}$.

Proof. $(i) \implies (ii)$ Assume that (i) holds. Let $\tilde{w}_\varsigma$ be a hybrid bi-ideal of $\mathscr{X}$ and n_1 be any element of $\mathscr{X}$. Since $\mathscr{X}$ is regular, there exists $l_1 \in \mathscr{X}$ such that $n_1 \leq n_1 l_1 n_1$. Now

$$\begin{aligned}
(\tilde{w} \circ \chi_{\mathscr{X}} \circ \tilde{w})(n_1) &= \bigcup_{n_1 \leq c_1 p_1} \{(\tilde{w} \circ \chi_{\mathscr{X}})(c_1) \cap \tilde{w}(p_1)\} \\
&\supseteq (\tilde{w} \circ \chi_{\mathscr{X}})(n_1 l_1) \cap \tilde{w}(n_1) \\
&= \bigcup_{n_1 l_1 \leq t_1 a_1} \{\tilde{w}(t_1) \cap \chi_{\mathscr{X}}(a_1)\} \cap \tilde{w}(n_1) \\
&\supseteq \tilde{w}(n_1) \cap \chi_{\mathscr{X}}(l_1) \cap \tilde{w}(n_1) = \tilde{w}(n_1); \\
(\varsigma \circ \chi_{\mathscr{X}} \circ \varsigma)(n_1) &= \bigwedge_{n_1 \leq c_1 p_1} \{(\varsigma \circ \chi_{\mathscr{X}})(c_1) \vee \varsigma(p_1)\} \\
&\leq (\varsigma \circ \chi_{\mathscr{X}})(n_1 l_1) \vee \varsigma(n_1) \\
&= \bigwedge_{n_1 l_1 \leq t_1 a_1} \{\varsigma(t_1) \vee \chi_{\mathscr{X}}(a_1)\} \vee \varsigma(n_1) \\
&\leq \varsigma(n_1) \vee \chi_{\mathscr{X}}(l_1) \vee \varsigma(n_1) = \varsigma(n_1).
\end{aligned}$$

Thus $\tilde{w}_\varsigma \ll \tilde{w}_\varsigma \odot \chi_{\mathscr{X}} \odot \tilde{w}_\varsigma$.

$(ii) \implies (iii)$ It follows directly from the proof of Theorem 3.3.

$(iii) \implies (i)$ Assume that (iii) holds and T is any quasi-ideal of $\mathscr{X}$. Then $\chi_T(\tilde{w}_\varsigma)$ of T is a hybrid quasi-ideal of $\mathscr{X}$. Now $\chi_T(\tilde{w}_\varsigma) \ll \chi_T(\tilde{w}_\varsigma) \odot \chi_{\mathscr{X}}(\tilde{w}_\varsigma) \odot \chi_T(\tilde{w}_\varsigma)$, we get $\chi_T(\tilde{w}_\varsigma) \ll \chi_{T\mathscr{X}T}(\tilde{w}_\varsigma)$. Therefore $T \subseteq T\mathscr{X}T$. Since T is a quasi-ideal of $\mathscr{X}$, we have $T\mathscr{X}T \subseteq \mathscr{X}T \cap T\mathscr{X} \subseteq T$ and so $T\mathscr{X}T = T$. Hence $\mathscr{X}$ is regular by Lemma 3.1. □

Theorem 3.7. *Let $\mathscr{X}$ be an ordered semiring. Then the below conditions are equivalent:*
(i) $\mathscr{X}$ is regular,
(ii) $\tilde{w}_\varsigma \Cap \tilde{b}_\daleth \ll \tilde{w}_\varsigma \odot \tilde{b}_\daleth \odot \tilde{w}_\varsigma$ for every hybrid bi-ideal $\tilde{w}_\varsigma$ and every hybrid ideal $\tilde{b}_\daleth$ of $\mathscr{X}$,
(iii) $\tilde{w}_\varsigma \Cap \tilde{b}_\daleth \ll \tilde{w}_\varsigma \odot \tilde{b}_\daleth \odot \tilde{w}_\varsigma$ for every hybrid quasi-ideal $\tilde{w}_\varsigma$ and every hybrid ideal $\tilde{b}_\daleth$ of $\mathscr{X}$.

Proof. $(i) \implies (ii)$ Assume that (i) holds. Let $\tilde{w}_\varsigma$ and $\tilde{b}_\daleth$ be the hybrid bi-ideal and hybrid ideal, respectively, of $\mathscr{X}$ and l_0 be any element of $\mathscr{X}$. Since $\mathscr{X}$ is regular, there exists $j_0 \in \mathscr{X}$ such that $l_0 \leq l_0 j_0 l_0$. Then

$$\begin{aligned}
(\tilde{w} \circ \tilde{b} \circ \tilde{w})(l_0) &= \bigcup_{l_0 \leq c_0 y_0} \{(\tilde{w} \circ \tilde{b})(c_0) \cap \tilde{w}(y_0)\} \\
&\supseteq (\tilde{w} \circ \tilde{b})(l_0 j_0) \cap \tilde{w}(l_0) \\
&= \bigcup_{l_0 j_0 \leq h_0 m_0} \{(\tilde{w}(h_0) \cap \tilde{b}(m_0))\} \cap \tilde{w}(l_0) \\
&\supseteq \tilde{w}(l_0) \cap \tilde{b}(j_0 l_0 j_0) \cap \tilde{w}(l_0) \supseteq \tilde{w}(l_0) \cap \tilde{b}(l_0) = (\tilde{w} \cap \tilde{b})(l_0); \\
(\varsigma \circ \daleth \circ \varsigma)(l_0) &= \bigwedge_{l_0 \leq c_0 y_0} \{(\varsigma \circ \daleth)(c_0) \vee \varsigma(y_0)\} \\
&\leq (\varsigma \circ \daleth)(l_0 j_0) \vee \varsigma(l_0) \\
&= \bigwedge_{l_0 j_0 \leq h_0 m_0} \{(\varsigma(h_0) \vee \daleth(m_0))\} \vee \varsigma(l_0) \\
&\leq \varsigma(l_0) \vee \daleth(j_0 l_0 j_0) \vee \varsigma(l_0) \leq \varsigma(l_0) \vee \daleth(l_0) = (\varsigma \vee \daleth)(l_0).
\end{aligned}$$

$(ii) \implies (iii)$ It follows directly from the proof of Theorem 3.3.

$(iii) \implies (i)$ Assume that (iii) holds. Let $\tilde{w}_\varsigma$ be a hybrid quasi-ideal of $\mathscr{X}$. Then, since $\chi_{\mathscr{X}}$ is a hybrid ideal of $\mathscr{X}$, we have $\tilde{w} = \tilde{w} \cap \chi_{\mathscr{X}} \subseteq \tilde{w} \circ \chi_{\mathscr{X}} \circ \tilde{w}$ and $\varsigma = \varsigma \vee \chi_{\mathscr{X}} \geq \varsigma \circ \chi_{\mathscr{X}} \circ \varsigma$. Hence $\mathscr{X}$ is regular by Theorem 3.6. □

4. Conclusion

This research primarily focuses on the concepts of hybrid bi-ideals and hybrid quasi-ideals within the framework of ordered semirings, examining their fundamental characteristics. Several equivalent criteria were established for an ordered semiring to be regular, utilizing the notions of hybrid bi-ideals, hybrid quasi-ideals, and hybrid ideals. Additionally, illustrative examples of hybrid bi-ideals and hybrid quasi-ideals were provided. In future investigations, we intend to introduce and study the notions of hybrid prime ideals, hybrid prime bi-ideals, and hybrid prime quasi-ideals, along with an analysis of their properties. Furthermore, the concepts proposed in this study may be extended to intuitionistic hybrid bi-ideals and intuitionistic hybrid quasi-ideals in ordered semirings, with potential applications to broader algebraic systems.

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