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SOME PROPERTIES OF FUZZY DISJOINTNESS IN FUZZY LATTICES

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ABSTRACT. In this paper, we introduce the concept of fuzzy general disjointness property along with its notation. Our results indicate that a fuzzy section semi-complemented lattice possesses the fuzzy atomic covering property if it meets the criteria for the fuzzy atomic disjointness property. Furthermore, we establish that if a fuzzy section semi-complemented lattice satisfies the fuzzy disjointness property, then it is both fuzzy $\perp$-modular and a fuzzy Birkhoff lattice.

Keywords: fuzzy lattices, fuzzy $\perp$-distributive, fuzzy $\perp$-modular, fuzzy general disjointness, fuzzy Birkhoff lattice, fuzzy atomic disjointness, fuzzy section semi-complemented lattice.

AMS Subject Classification: 03B52, 03E72, 06D72, 06D99.

1. Introduction

Many researchers have investigated the generalization of distributivity, modularity, and semi-modularity in various mathematical structures. Saarimäki and Sorjonen [18] defined concepts such as 0-distributivity, the disjointness property, and the atomic disjointness property within the context of a lattice. Waphare and Joshi [21] introduced the notions of 0-distributivity and 0-modularity in partially ordered sets (posets), along with a general definition of the disjointness property for posets, and explored a variety of related properties.

The concept of a fuzzy set was introduced by Zadeh [27], leading to numerous researchers proposing different ideas associated with fuzzy algebraic structures. For instance, Rosenfeld [17] introduced fuzzy groups, while Ajmal et al. [1] and Chon [2] developed the concept of fuzzy lattices. Also defined fuzzy modular (A fuzzy lattice (X, A) is called

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fuzzy modular if $A(c,b) > 0$ implies $c \vee_F (a \wedge_F b) = (c \vee_F a) \wedge_F b$ for $a, b, c \in X$) which has also been examined by Mezzomo et. al. [10, 11, 12].

Wasadikar and Khubchandani [22] put forward the idea of a fuzzy modular pair within a fuzzy lattice. Moreover in [24] they defined fuzzy Birkhoff lattice(A fuzzy lattice $\mathcal{L} = (X, A)$ is called a fuzzy Birkhoff lattice if in $\mathcal{L}$, $a \wedge_F b \prec_F a, b$ implies that $a, b \prec_F a \vee_F b$ for $a, b \in X$). In [26] they defined fuzzy atomic property (A fuzzy lattice (X, A) with $\bot$ is called fuzzy atomic property if for $a, b \in X$ with $a \wedge_F b \neq \bot$, then there exists a fuzzy atom $p \in X$ such that $\bot \prec_F p$ and $A(p, a \wedge_F b) > 0$). In addition to this, they defined fuzzy atomistic (A fuzzy poset (X, A) with the least element $\bot$ is called fuzzy atomistic if every element $x \in X$ is the least upper bound of the set of fuzzy atoms less than or equal to x).

Additionally, Khubchandani and Khubchandani [4] introduced the concepts and notation for fuzzy $\bot$-distributive lattices and their corresponding dual concept of fuzzy $\top$-distributive lattices. They demonstrated that a fuzzy section semicomplemented lattice is fuzzy distributive if and only if it is fuzzy $\bot$-distributive. Furthermore, they proved that a fuzzy pseudo-complemented lattice is also a fuzzy $\bot$-distributive lattice.

The motivation behind this paper is to further explore these concepts by building on the innovative and insightful ideas of Khubchandani and Khubchandani [4]. The objective is to enhance the existing understanding of the topic and contribute meaningfully to the field. This current paper highlights the significance of Khubchandani's work and how it has inspired further research.

In section 3, we introduce several new notions, including the fuzzy general disjointness property, fuzzy atomic disjointness property, fuzzy atomic exchange property, and fuzzy atomic covering property. Additionally, we have proved that a fuzzy $\bot$-modular lattice possesses the fuzzy disjointness property.

In Section 4, we demonstrate that if a lattice exhibits the fuzzy atomic disjointness property, then it inherently possesses the fuzzy atomic covering property. Moreover, if (X, A) is fuzzy atomic, it can be classified as a fuzzy Birkhoff lattice.

This research makes a significant theoretical contribution to the field of fuzzy algebraic structures by generalizing and refining classical properties within the fuzzy context. The introduction of new properties and the derivation of interrelationships between them deepen our understanding of fuzzy lattices and posets, and highlight a growing internal coherence within fuzzy systems.

These findings are particularly relevant in domains where fuzzy logic and non-binary reasoning are crucial, such as decision theory, fuzzy information systems, artificial intelligence, and approximate reasoning. By generalizing classical concepts like modularity and atomicity to fuzzy settings, this work enables more flexible and expressive models of uncertainty—an essential requirement in many modern computational and analytical applications.

In essence, this research contributes to the establishment of a comprehensive and internally consistent algebraic framework for fuzzy structures, offering valuable insights and tools for both theoretical exploration and practical implementation.

2. Preliminaries

Throughout in this paper, $(X,\ A)$ denotes a fuzzy lattice, where A is a fuzzy partial order relation on a non empty set X.

For the definitions of a fuzzy partial order relation, fuzzy equivalence relation, fuzzy supremum, fuzzy infimum, fuzzy lattice etc. we refer to Chon [2]. We use the notations $a \vee_F b$

and $a \wedge_F b$ to denote the fuzzy supremum and the fuzzy infimum of $a, b \in X$ to distinguish the supremum and infimum of a, b in the lattice sense, if these exist in X.

We recall some known results from Chon [2] which we shall use in this paper.

Proposition 2.1. [2, Proposition 3.3] *and* [9, Proposition 2.4] *Let (X, A) be a fuzzy lattice. For $a, b, c \in X$. The following statements hold:*
(i) $A(a, c) > 0$ and $A(b, c) > 0$ implies $A(a \vee_F b, c) > 0$;
(ii) $A(a, b) > 0$ iff $a \vee_F b = b$;
(iii) $A(a, b) > 0$ iff $a \wedge_F b = a$.

Definition 2.1. *(Chon* [2]*) Let (X, A) be a fuzzy lattice. (X, A) is called a fuzzy distributive lattice, if $a \wedge_F (b \vee_F c) = (a \wedge_F b) \vee_F (a \wedge_F c)$ and $a \vee_F (b \wedge_F c) = (a \vee_F b) \wedge_F (a \vee_F c)$ for all $a, b, c \in X$.*

We recall some definitions from Khubchandani and Khubchandani [3, 4]

Definition 2.2. [3, Definition 3.1] *A fuzzy lattice (X, A) is called fuzzy section semi-complemented lattice $(FSSC)$ if it satisfies the following condition:*
If $a \neq b$ in X, then there exists $c \in X$ such that $c \neq \perp$, $A(c, a) > 0$ and $c \wedge_F b = \perp$.

Definition 2.3. [4, Definition 3.1] *A fuzzy lattice (X, A) with $\perp$ is called fuzzy $\perp$-distributive lattice if $a \wedge_F b = \perp$, $a \wedge_F c = \perp$ together imply $a \wedge_F (b \vee_F c) = \perp$, for all $a, b, c \in X$.*

Definition 2.4. [4, Definition 3.2] *A fuzzy lattice (X, A) with $\top$ is called fuzzy $\top$-distributive lattice if $a \vee_F b = \top$, $a \vee_F c = \top$ together imply $a \vee_F (b \wedge_F c) = \top$, for all $a, b, c \in X$.*

Definition 2.5. [4, Definition 3.3] *A fuzzy bounded lattice $\mathcal{L} = (X, A)$ which is both fuzzy $\perp$-distributive lattice and fuzzy $\top$-distributive lattice is called fuzzy $\perp$-$\top$ distributive lattice.*

Definition 2.6. [10, Definition 3.4] *A fuzzy lattice $\mathcal{L} = (X, A)$ is bounded if there exist elements $\perp$ and $\top$ in X, such that $A(\perp, a) > 0$ and $A(a, \top) > 0$, for all $a \in X$. In this case, $\perp$ and $\top$ are called bottom and top elements, respectively.*
A fuzzy bounded lattice (X, A) is said called fuzzy complemented if every element $a \in X$ has a complement $b \in X$ with $a \wedge_F b = \perp$ and $a \vee_F b = \top$.

Definition 2.7. [22, Definition 3.3] *Let P denote the set of all $a \in X$ such that $\perp \prec_F a$. The elements of P are called fuzzy atoms.*

Definition 2.8. [22, Definition 4.4] *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice. Let a, $b \in X$. We say that a fuzzy covers b and write $b \prec_F a$, if $0 < A(b, a) < 1$ and $A(b, c) > 0$ and $A(c, a) > 0$ imply $c = a$ or $c = b$.*

Lemma 2.1. [22, Lemma 4.2] *If $a \prec_F a \vee_F b$ and $(a, b)_F M$ then $a \wedge_F b \prec_F b$.*

Corollary 2.1. [25, Corollary 3] *Let (X, A) be a fuzzy lattice and $a, b, c, d \in X$. If $A(c, a) > 0$ and $A(d, b) > 0$, then $A(c \wedge_F d, a \wedge_F b) > 0$ and $A(c \vee_F d, a \vee_F b) > 0$.*

3. Fuzzy Disjointness Properties in Fuzzy Lattices

Wasadikar et. al. [22] and Khubchandani et. al. [4] have proved results related to fuzzy modular and fuzzy $\perp$-distributive lattice. Their work has motivated us to introduce and study fuzzy general disjointness property in fuzzy lattice.

Definition 3.1. *A fuzzy lattice (X, A) with $\perp$ is said to have fuzzy general disjointness property (FGD), if $a \wedge_F b = \perp$ and $(a \vee_F b) \wedge_F c = \perp$, then $a \wedge_F (b \vee_F c) = \perp$, for all $a, b, c \in X$.*

Example 3.1. *Let* $X = \{\bot, a, b, c, d, e, \top\}$. *Define a fuzzy relation* $A : X \times X \longrightarrow [0, 1]$ *on* X *as follows such that*
$A(\bot,\bot) = A(a,a) = A(b,b) = A(c,c) = A(d,d) = A(e,e) = A(\top,\top) = 1,$
$A(\bot,a) = 0.3,\ A(\bot,b) = 0.3,\ A(\bot,c) = 0.3,\ A(\bot,d) = 0.3,\ A(\bot,e) = 0.3,\ A(\bot,\top) = 0.3,$
$A(a,\bot) = 0,\ A(a,b) = 0,\ A(a,c) = 0,\ A(a,d) = 0.6,\ A(a,e) = 0, A(a,\top) = 0.03,$
$A(b,\bot) = 0,\ A(b,a) = 0,\ A(b,c) = 0,\ A(b,d) = 0.6,\ A(b,e) = 0.6,\ A(b,\top) = 0.03,$
$A(c,\bot) = 0,\ A(c,a) = 0,\ A(c,b) = 0,\ A(c,d) = 0,\ A(c,e) = 0.6,\ A(c,\top) = 0.03,$
$A(d,\bot) = 0,\ A(d,a) = 0,\ A(d,b) = 0,\ A(d,c) = 0,\ A(d,e) = 0,\ A(d,\top) = 0.03,$
$A(e,\bot) = 0,\ A(e,a) = 0,\ A(e,b) = 0,\ A(e,c) = 0,\ A(e,d) = 0,\ A(e,\top) = 0.03,$
$A(\top,\bot) = 0,\ A(\top,a) = 0,\ A(\top,b) = 0,\ A(\top,c) = 0,\ A(\top,d) = 0,\ A(\top,e) = 0.$

This fuzzy relation is shown in the following table:

A	$\bot$	a	b	c	d	e	$\top$
$\bot$	1.0	0.3	0.3	0.3	0.3	0.3	0.3
a	0.0	1.0	0.0	0.0	0.6	0.0	0.03
b	0.0	0.0	1.0	0.0	0.6	0.6	0.03
c	0.0	0.0	0.0	1.0	0.0	0.6	0.03
d	0.0	0.0	0.0	0.0	1.0	0.0	0.03
e	0.0	0.0	0.0	0.0	0.0	1.0	0.03
$\top$	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	a	b	c	d	e	$\top$
a	a	a	d	$\top$	d	$\top$	$\top$
b	b	d	b	e	d	e	$\top$
c	c	$\top$	e	c	$\top$	e	$\top$
d	d	d	d	$\top$	d	$\top$	$\top$
e	e	$\top$	e	e	$\top$	e	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a	$\bot$	a
b	$\bot$	$\bot$	b	$\bot$	b	b	b
c	$\bot$	$\bot$	$\bot$	c	$\bot$	c	c
d	$\bot$	a	b	$\bot$	d	b	d
e	$\bot$	$\bot$	b	c	b	e	e
$\top$	$\bot$	a	b	c	d	e	$\top$

We note that (X, A) is a fuzzy lattice.
(X, A) satisfies a fuzzy general disjointness property (FGD) as $a \wedge_F b = \bot$, $(a \vee_F b) \wedge_F c = d \wedge_F c = \bot$ then $a \wedge_F (b \vee_F c) = a \wedge_F e = \bot$.

Definition 3.2. *A fuzzy lattice* (X, A) *with* $\bot$ *is said to have fuzzy atomic disjointness property (FAD) if for fuzzy atom* $p \in X$, $\bot \prec_F p$, $a \wedge_F b = \bot$ *and* $(a \vee_F b) \wedge_F p = \bot$, *then* $a \wedge_F (b \vee_F p) = \bot$ *for all* $a, b \in X$.

Example 3.2. *Let* $X = \{\bot, a, b, c, d, e, f, \top\}$. *Define a fuzzy relation* $A : X \times X \longrightarrow [0, 1]$ *on* X *as follows such that*
$A(\bot,\bot) = A(a,a) = A(b,b) = A(c,c) = A(d,d) = A(e,e) = A(f,f) = A(\top,\top) = 1,$
$A(\bot,a) = 0.5,\ A(\bot,b) = 0.5,\ A(\bot,c) = 0.5,\ A(\bot,d) = 0.5,\ A(\bot,e) = 0.5,\ A(\bot,f) = 0.5,$
$A(\bot,\top) = 0.5,$
$A(a,\bot) = 0,\ A(a,b) = 0,\ A(a,c) = 0,\ A(a,d) = 0.8,\ A(a,e) = 0.8,\ A(a,f) = 0,$
$A(a,\top) = 0.05,$
$A(b,\bot) = 0,\ A(b,a) = 0,\ A(b,c) = 0,\ A(b,d) = 0,\ A(b,e) = 0,\ A(b,f) = 0,\ A(b,\top) = 0.05,$
$A(c,\bot) = 0,\ A(c,a) = 0,\ A(c,b) = 0.8,\ A(c,d) = 0,\ A(c,e) = 0,\ A(c,f) = 0,$
$A(c,\top) = 0.05,$
$A(d,\bot) = 0,\ A(d,a) = 0,\ A(d,b) = 0,\ A(d,c) = 0,\ A(d,e) = 0.8,\ A(d,f) = 0,$
$A(d,\top) = 0.05,$
$A(e,\bot) = 0,\ A(e,a) = 0,\ A(e,b) = 0,\ A(e,c) = 0,\ A(e,d) = 0,\ A(e,f) = 0,$

$A(e,\top) = 0.05,$
$A(f,\bot) = 0,\ A(f,a) = 0,\ A(f,b) = 0,\ A(f,c) = 0.8,\ A(f,d) = 0.8,\ A(f,e) = 0.8,$
$A(f,\top) = 0.05,$
$A(\top,\bot) = 0,\ A(\top,a) = 0,\ A(\top,b) = 0,\ A(\top,c) = 0,\ A(\top,d) = 0,\ A(\top,e) = 0,$
$A(\top,f) = 0.$

This fuzzy relation is shown in the following table:

A	$\bot$	a	b	c	d	e	f	$\top$
$\bot$	1.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5
a	0.0	1.0	0.0	0.0	0.8	0.8	0.0	0.05
b	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.05
c	0.0	0.0	0.8	1.0	0.0	0.0	0.0	0.05
d	0.0	0.0	0.0	0.0	1.0	0.8	0.0	0.05
e	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.05
f	0.0	0.0	0.0	0.8	0.8	0.8	1.0	0.05
$\top$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	d	e	f	$\top$
$\bot$	$\bot$	a	b	c	d	e	f	$\top$
a	a	a	$\top$	$\top$	d	e	d	$\top$
b	b	$\top$	b	c	$\top$	$\top$	c	$\top$
c	c	$\top$	c	c	$\top$	$\top$	c	$\top$
d	d	d	$\top$	$\top$	d	e	d	$\top$
e	e	e	$\top$	$\top$	e	e	e	$\top$
f	f	d	c	c	d	e	f	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	d	e	f	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a	a	$\bot$	a
b	$\bot$	$\bot$	b	b	$\bot$	$\bot$	$\bot$	b
c	$\bot$	$\bot$	b	c	f	f	f	c
d	$\bot$	a	$\bot$	f	d	d	f	d
e	$\bot$	a	$\bot$	f	d	e	f	e
f	$\bot$	$\bot$	$\bot$	f	f	f	f	f
$\top$	$\bot$	a	b	c	d	e	f	$\top$

We note that (X, A) is a fuzzy lattice.
(X, A) satisfies fuzzy atomic disjointness property (FAD) as $\bot \prec_F b$, $a \wedge_F f = \bot$, $(a \vee_F f) \wedge_F b = d \wedge_F b = \bot$ implies $a \wedge_F (f \vee_F b) = a \wedge_F c = \bot$.

Definition 3.3. *A fuzzy lattice (X, A) with $\bot$ is said to have fuzzy atomic exchange property (FAE) if for fuzzy atoms $p, q \in X$, $\bot \prec_F p$, $\bot \prec_F q$, $a \wedge_F p = \bot$, $A(p, a \vee_F q) > 0$, then $A(q, a \vee_F p) > 0$, for $a \in X$.*

Example 3.3. *Let $X = \{\bot, a, b, c, d, e, f, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0,1]$ on X as follows such that*
$A(\bot,\bot) = A(a,a) = A(b,b) = A(c,c) = A(d,d) = A(e,e) = A(f,f) = A(\top,\top) = 1,$
$A(\bot,a) = 0.7,\ A(\bot,b) = 0.7,\ A(\bot,c) = 0.7,\ A(\bot,d) = 0.7,\ A(\bot,e) = 0.7,\ A(\bot,f) = 0.7,$
$A(\bot,\top) = 0.7,$
$A(a,\bot) = 0,\ A(a,b) = 0,\ A(a,c) = 0,\ A(a,d) = 0.9,\ A(a,e) = 0,\ A(a,f) = 0,$
$A(a,\top) = 0.07,$
$A(b,\bot) = 0,\ A(b,a) = 0,\ A(b,c) = 0,\ A(b,d) = 0.9,\ A(b,e) = 0.9,\ A(b,f) = 0,$
$A(b,\top) = 0.07,$
$A(c,\bot) = 0,\ A(c,a) = 0,\ A(c,b) = 0,\ A(c,d) = 0,\ A(c,e) = 0,\ A(c,f) = 0.9,$
$A(c,\top) = 0.07,$
$A(d,\bot) = 0,\ A(d,a) = 0,\ A(d,b) = 0,\ A(d,c) = 0,\ A(d,e) = 0,\ A(d,f) = 0,$
$A(d,\top) = 0.07,$
$A(e,\bot) = 0,\ A(e,a) = 0,\ A(e,b) = 0,\ A(e,c) = 0,\ A(e,d) = 0,\ A(e,f) = 0,$
$A(e,\top) = 0.07,$
$A(f,\bot) = 0,\ A(f,a) = 0,\ A(f,b) = 0,\ A(f,c) = 0,\ A(f,d) = 0,\ A(f,e) = 0,$

$A(f, \top) = 0.07$,
$A(\top, \bot) = 0$, $A(\top, a) = 0$, $A(\top, b) = 0$, $A(\top, c) = 0$, $A(\top, d) = 0$, $A(\top, e) = 0$,
$A(\top, f) = 0$.

This fuzzy relation is shown in the following table:

A	$\bot$	a	b	c	d	e	f	$\top$
$\bot$	1.0	0.7	0.7	0.7	0.7	0.7	0.7	0.7
a	0.0	1.0	0.0	0.0	0.9	0.0	0.0	0.07
b	0.0	0.0	1.0	0.0	0.9	0.9	0.0	0.07
c	0.0	0.0	0.0	1.0	0.0	0.0	0.9	0.07
d	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.07
e	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.07
f	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.07
$\top$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	d	e	f	$\top$
$\bot$	$\bot$	a	b	c	d	e	f	$\top$
a	a	a	d	$\top$	d	$\top$	$\top$	$\top$
b	b	d	b	$\top$	d	e	$\top$	$\top$
c	c	$\top$	$\top$	c	$\top$	$\top$	f	$\top$
d	d	d	d	$\top$	d	$\top$	$\top$	$\top$
e	e	$\top$	e	$\top$	$\top$	e	$\top$	$\top$
f	f	$\top$	$\top$	f	$\top$	$\top$	f	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	d	e	f	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a	$\bot$	$\bot$	a
b	$\bot$	$\bot$	b	$\bot$	b	b	$\bot$	b
c	$\bot$	$\bot$	$\bot$	c	$\bot$	$\bot$	c	c
d	$\bot$	a	b	$\bot$	d	b	$\bot$	d
e	$\bot$	$\bot$	b	$\bot$	b	e	$\bot$	e
f	$\bot$	$\bot$	$\bot$	c	$\bot$	$\bot$	f	f
$\top$	$\bot$	a	b	c	d	e	f	$\top$

We note that (X, A) is a fuzzy lattice.
(X, A) satisfies fuzzy atomic exchange property (FAE) as $\bot \prec_F c$, $\bot \prec_F a$, $e \wedge_F c = \bot$ and $A(c, e \vee_F a) = A(c, \top) = 0.07 > 0$ implies $A(a, e \vee_F c) = A(a, \top) = 0.07 > 0$.

Definition 3.4. *A fuzzy lattice (X, A) with $\bot$ is said to have fuzzy atomic covering property (FAC) if for fuzzy atom $p \in X$, $\bot \prec_F p$ and $a \wedge_F p = \bot$, then $a \prec_F a \vee_F p$ for $a \in X$.*

Example 3.4. *Consider the fuzzy lattice in example 3.1 as $\bot \prec_F a$, $a \wedge_F b = \bot$ implies $a \prec_F a \vee_F b$ that is $a \prec_F d$. Therefore fuzzy atomic covering property (FAC) is satisfied.*

Remark 3.1. *Fuzzy atomic disjoint property is implied by the fuzzy atomic covering which in turns fulfilled in fuzzy Birkhoff lattice property.*

An immediate consequence of the fuzzy modularity is the following result.

Definition 3.5. *Let (X, A) be a fuzzy lattice with $\bot$. (X, A) is said to be fuzzy $\bot$-modular if for $a, b, c \in X$, $a \wedge_F b = \bot$ and $A(c, b) > 0$ together imply $(c \vee_F a) \wedge_F b = c$.*

Dually, we have the concept of fuzzy $\top$-modular lattice.

Lemma 3.1. *Every fuzzy modular lattice is fuzzy $\bot$-modular lattice.*

Proof. Proof follows from the definition. □

In the following example we show that the converse of above Lemma 3.1 is not true.

Example 3.5. *Let $X = \{\bot, a, b, c, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\bot, \bot) = A(a, a) = A(b, b) = A(c, c) = A(\top, \top) = 1$,

$A(\bot, a) = 0.6$, $A(\bot, b) = 0.6$, $A(\bot, c) = 0.6$, $A(\bot, \top) = 0.6$,
$A(a, \bot) = 0$, $A(a, b) = 0$, $A(a, c) = 0$, $A(a, \top) = 0.04$,
$A(b, \bot) = 0$, $A(b, a) = 0$, $A(b, c) = 0.3$, $A(b, \top) = 0.04$,
$A(c, \bot) = 0$, $A(c, a) = 0$, $A(c, b) = 0$, $A(c, \top) = 0.04$,
$A(\top, \bot) = 0$, $A(\top, a) = 0$, $A(\top, b) = 0$, $A(\top, c) = 0$.

This fuzzy relation is shown in the following table:

A	$\bot$	a	b	c	$\top$
$\bot$	1.0	0.6	0.6	0.6	0.6
a	0.0	1.0	0.0	0.0	0.04
b	0.0	0.0	1.0	0.3	0.04
c	0.0	0.0	0.0	1.0	0.04
$\top$	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	a	b	c	$\top$
a	a	a	$\top$	$\top$	$\top$
b	b	$\top$	b	c	$\top$
c	c	$\top$	c	c	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a
b	$\bot$	$\bot$	b	b	b
c	$\bot$	$\bot$	b	c	c
$\top$	$\bot$	a	b	c	$\top$

We note that (X, A) is a fuzzy lattice.
As $a \wedge_F c = \bot$, $A(b, c) > 0$ and $(b \vee_F a) \wedge_F c = b$.
Hence (X, A) is fuzzy $\bot$-modular lattice.
As $A(b, c) > 0$ but $(b \vee_F a) \wedge_F c = c \neq b \vee_F (a \wedge_F c) = b$.
Hence (X, A) is not fuzzy modular lattice.
Therefore (X, A) is Fuzzy $\bot$-modular lattice but not fuzzy modular lattice.

Remark 3.2. *A fuzzy modular lattice (X, A) with $\bot$ also satisfies the fuzzy general disjointness property.*

In the Remark below we show that a weaker assumption is sufficient.

Remark 3.3. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\bot$. Let $a, b \in X$. We say (a, b) is a fuzzy $\bot$-modular pair if every disjoint pair (a, b) is fuzzy modular or in other words, if it satisfies the condition $a \wedge_F b = \bot$, $A(c, b) > 0$ implies $c = (a \vee_F c) \wedge_F b$.*

Theorem 3.1. *Let (X, A) be a fuzzy lattice with $\bot$. A fuzzy $\bot$-modular lattice has the fuzzy general disjointness property.*

Proof. Let (X, A) be a $\bot$-modular lattice and let a, b, c be elements of X with $a \wedge_F b = \bot$ and $(a \vee_F b) \wedge_F c = \bot$.
If we denote $d = (a \vee_F b) \wedge_F (b \vee_F c)$ then by (iii) of Proposition 2.1 we have $A(b, d) > 0$ and $A(d, b \vee_F c) > 0$. Taking join c on both sides we get

$$A(b \vee_F c, d \vee_F c) > 0 \tag{1}$$

and

$$A(d \vee_F c, b \vee_F c) > 0. \tag{2}$$

From (1) and (2) by fuzzy antisymmetry of A we get $b \vee_F c = d \vee_F c$.
On the other hand $A(d, a \vee_F b) > 0$.
Taking meet c on both sides we get $A(d \wedge_F c, (a \vee_F b) \wedge_F c) > 0$, i.e.,

$$A(d \wedge_F c, \bot) > 0. \tag{3}$$

And

$$A(\bot, d \wedge_F c) > 0 \tag{4}$$

always holds.
From (3) and (4) by fuzzy antisymmetry of A we get $d \wedge_F c = \bot$.
By assumption (c, d) is fuzzy $\bot$-modular pair, so that $b = (b \vee_F c) \wedge_F d = d$.
Finally we get

$$a \wedge_F (b \vee_F c) = a \wedge_F d = a \wedge_F b = \bot$$

which proves the assertion. □

Proposition 3.1. *If (X, A) is a fuzzy lattice with $\bot$ satisfies FAD, then it satisfies FAE. In fuzzy atomic lattice FAD and FAE are equivalent.*

Proof. Let (X, A) satisfies FAD. Assume that $a \wedge_F p = \bot$ and $A(p, a \vee_F q) > 0$ for fuzzy atoms $p, q \in X$. If $(a \vee_F p) \wedge_F q = \bot$, then $\bot = p \wedge_F (a \vee_F q) = p$ by FAD, which is impossible. Hence $A(q, a \vee_F p) > 0$ and thus the condition FAE is fulfilled.

Assume (X, A) is fuzzy atomic and satisfies FAE. Let $a \wedge_F b = \bot$ and $(a \vee_F b) \wedge_F p = \bot$ for a fuzzy atom $p \in X$. If $a \wedge_F (b \vee_F p) \neq \bot$, then there exists a fuzzy atom $q \in X$ such that $A(q, a \wedge_F (b \vee_F p)) > 0$. Since $A(q, b \vee_F p) > 0$ and $b \wedge_F p = \bot$, then $A(p, b \vee_F q) > 0$ by FAE and so $A(p, b \vee_F a) > 0$, a contradiction. Hence $a \wedge_F (b \vee_F p) = \bot$ and FAD is satisfied. □

In fuzzy lattice (X, A) with least element $\bot$. The following implications hold:

$$Fuzzy\ Birkhoff \Rightarrow FAC \Rightarrow FAD \Rightarrow FAE.$$

The converses of the first two implications do not hold even in atomistic lattices, counterexamples are easy to find.

The following example shows that a fuzzy atomistic lattice, need not satisfy FGD if it satisfies FAD.

Example 3.6. *Let $X = \{\bot, a, b, c, d, e, f, g, h, i, j, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\bot, \bot) = A(a, a) = A(b, b) = A(c, c) = A(d, d) = A(e, e) = A(f, f) = 1$,
$A(g, g) = A(h, h) = A(i, i) = A(j, j) = A(\top, \top) = 1$,
$A(\bot, a) = 0.4,\ A(\bot, b) = 0.4,\ A(\bot, c) = 0.4,\ A(\bot, d) = 0.4,\ A(\bot, e) = 0.4,\ A(\bot, f) = 0.4$,
$A(\bot, g) = 0.4,\ A(\bot, h) = 0.4,\ A(\bot, i) = 0.4,\ A(\bot, j) = 0.4,\ A(\bot, \top) = 0.4$,
$A(a, \bot) = 0,\ A(a, b) = 0,\ A(a, c) = 0,\ A(a, d) = 0,\ A(a, e) = 0.8,\ A(a, f) = 0.8$,
$A(a, g) = 0.8,\ A(a, h) = 0,\ A(a, i) = 0,\ A(a, j) = 0,\ A(a, \top) = 0.04$,
$A(b, \bot) = 0,\ A(b, a) = 0,\ A(b, c) = 0,\ A(b, d) = 0,\ A(b, e) = 0.8,\ A(b, f) = 0,\ A(b, g) = 0$,
$A(b, h) = 0.8,\ A(b, i) = 0.8,\ A(b, j) = 0,\ A(b, \top) = 0.04$,
$A(c, \bot) = 0,\ A(c, a) = 0,\ A(c, b) = 0,\ A(c, d) = 0,\ A(c, e) = 0,\ A(c, f) = 0.8,\ A(c, g) = 0$,
$A(c, h) = 0.8,\ A(c, i) = 0,\ A(c, j) = 0.8,\ A(c, \top) = 0.04$,
$A(d, \bot) = 0,\ A(d, a) = 0,\ A(d, b) = 0,\ A(d, c) = 0,\ A(d, e) = 0,\ A(d, f) = 0,\ A(d, g) = 0.8$,
$A(d, h) = 0,\ A(d, i) = 0.8,\ A(d, j) = 0.8,\ A(d, \top) = 0.04$,
$A(e, \bot) = 0,\ A(e, a) = 0,\ A(e, b) = 0,\ A(e, c) = 0,\ A(e, d) = 0,\ A(e, f) = 0,\ A(e, g) = 0$,
$A(e, h) = 0,\ A(e, i) = 0,\ A(e, j) = 0,\ A(e, \top) = 0.04$,
$A(f, \bot) = 0,\ A(f, a) = 0,\ A(f, b) = 0,\ A(f, c) = 0,\ A(f, d) = 0,\ A(f, e) = 0,\ A(f, g) = 0$,
$A(f, h) = 0,\ A(f, i) = 0,\ A(f, j) = 0,\ A(f, \top) = 0.04$,
$A(g, \bot) = 0,\ A(g, a) = 0,\ A(g, b) = 0,\ A(g, c) = 0,\ A(g, d) = 0,\ A(g, e) = 0,\ A(g, f) = 0$,
$A(g, h) = 0,\ A(g, i) = 0,\ A(g, j) = 0,\ A(g, \top) = 0.04$

$A(h,\bot) = 0$, $A(h,a) = 0$, $A(h,b) = 0$, $A(h,c) = 0$, $A(h,d) = 0$, $A(h,e) = 0$, $A(h,f) = 0$, $A(h,g) = 0$, $A(h,i) = 0$, $A(h,j) = 0$, $A(h,\top) = 0.04$,
$A(i,\bot) = 0$, $A(i,a) = 0$, $A(i,b) = 0$, $A(i,c) = 0$, $A(i,d) = 0$, $A(i,e) = 0$, $A(i,f) = 0$, $A(i,g) = 0$, $A(i,h) = 0$, $A(i,j) = 0$, $A(i,\top) = 0.04$,
$A(j,\bot) = 0$, $A(j,a) = 0$, $A(j,b) = 0$, $A(j,c) = 0$, $A(j,d) = 0$, $A(j,e) = 0$, $A(j,f) = 0$, $A(j,g) = 0$, $A(j,h) = 0$, $A(j,i) = 0$, $A(j,\top) = 0.04$,
$A(\top,\bot) = 0$, $A(\top,a) = 0$, $A(\top,b) = 0$, $A(\top,c) = 0$, $A(\top,d) = 0$, $A(\top,e) = 0$, $A(\top,f) = 0$, $A(\top,g) = 0$, $A(\top,h) = 0$,$A(\top,i) = 0$, $A(\top,j) = 0$.

This fuzzy relation is shown in the following table:

A	⊥	a	b	c	d	e	f	g	h	i	j	⊤
⊥	1.0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
a	0.0	1.0	0.0	0.0	0.0	0.8	0.8	0.8	0.0	0.0	0.0	0.04
b	0.0	0.0	1.0	0.0	0.0	0.8	0.0	0.0	0.8	0.8	0.0	0.04
c	0.0	0.0	0.0	1.0	0.0	0.0	0.8	0.0	0.8	0.0	0.8	0.04
d	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.8	0.0	0.8	0.8	0.04
e	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.04
f	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.04
g	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.04
h	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.04
i	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.04
j	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.04
⊤	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	⊥	a	b	c	d	e	f	g	h	i	j	⊤
⊥	⊥	a	b	c	d	e	f	g	h	i	j	⊤
a	a	a	e	f	g	e	f	g	⊤	⊤	⊤	⊤
b	b	e	b	h	i	e	⊤	⊤	h	i	⊤	⊤
c	c	f	h	c	j	⊤	f	⊤	h	⊤	j	⊤
d	d	g	i	j	d	⊤	⊤	g	⊤	i	j	⊤
e	e	e	e	⊤	⊤	e	⊤	⊤	⊤	⊤	⊤	⊤
f	f	f	⊤	f	⊤	⊤	f	⊤	⊤	⊤	⊤	⊤
g	g	g	⊤	⊤	g	⊤	⊤	g	⊤	⊤	⊤	⊤
h	h	⊤	h	h	⊤	⊤	⊤	⊤	h	⊤	⊤	⊤
i	i	⊤	i	⊤	i	⊤	⊤	⊤	⊤	i	⊤	⊤
j	j	j	⊤	j	j	⊤	⊤	⊤	⊤	⊤	j	⊤
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤

$\wedge_F$	⊥	a	b	c	d	e	f	g	h	i	j	⊤
⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
a	⊥	a	⊥	⊥	⊥	a	a	a	⊥	⊥	⊥	a
b	⊥	⊥	b	⊥	⊥	b	⊥	⊥	b	b	⊥	b
c	⊥	⊥	⊥	c	⊥	⊥	c	⊥	c	⊥	c	c
d	⊥	⊥	⊥	⊥	d	⊥	⊥	d	⊥	d	d	d
e	⊥	a	b	⊥	⊥	e	a	a	⊥	⊥	⊥	e
f	⊥	a	⊥	c	⊥	a	f	a	⊥	⊥	⊥	f
g	⊥	a	⊥	⊥	d	a	a	g	⊥	⊥	d	g
h	⊥	⊥	b	c	⊥	⊥	⊥	⊥	h	b	c	h
i	⊥	⊥	b	⊥	d	⊥	⊥	⊥	b	i	d	i
j	⊥	⊥	⊥	c	d	⊥	⊥	d	c	d	j	j
⊤	⊥	a	b	c	d	e	f	g	h	i	j	⊤

We note that (X, A) is a fuzzy atomistic lattice.
As $a \wedge_F c = \bot$ and $(a \vee_F c) \wedge_F i = f \wedge_F i = \bot$.
But $a \wedge_F (c \vee_F i) = a \wedge_F \top = a \neq \bot$.
Therefore FGD property is not satisfied.
As $\bot \prec_F b$, $a \wedge_F c = \bot$ and $(a \vee_F c) \wedge_F b = f \wedge_F b = \bot$ implies $a \wedge_F (c \vee_F b) = a \wedge_F h = \bot$.
Therefore FAD property is satisfied.
We conclude that fuzzy atomistic lattice need not satisfy FGD if it satisfies FAD.

In the following example we show that a fuzzy lattice (X, A) satisfies FGD but does not satisfy FAC.

Example 3.7. *Let $X = \{\bot, a, b, c, d, e, f, g, h, i, j, \top\}$. Define a fuzzy relation $A : X \times X \longrightarrow [0, 1]$ on X as follows such that*
$A(\bot,\bot) = A(a,a) = A(b,b) = A(c,c) = A(d,d) = A(e,e) = A(f,f) = 1$,
$A(g,g) = A(h,h) = A(i,i) = A(j,j) = A(\top,\top) = 1$,

$A(\bot, a) = 0.2,\ A(\bot, b) = 0.2,\ A(\bot, c) = 0.2,\ A(\bot, d) = 0.2,\ A(\bot, e) = 0.2,\ A(\bot, f) = 0.2,$
$A(\bot, g) = 0.2,\ A(\bot, h) = 0.2,\ A(\bot, i) = 0.2,\ A(\bot, j) = 0.2,\ A(\bot, \top) = 0.2,$
$A(a, \bot) = 0,\ A(a, b) = 0,\ A(a, c) = 0,\ A(a, d) = 0.6,\ A(a, e) = 0.6,\ A(a, f) = 0,$
$A(a, g) = 0,\ A(a, h) = 0.6,\ A(a, i) = 0.6,\ A(a, j) = 0,\ A(a, \top) = 0.02,$
$A(b, \bot) = 0,\ A(b, a) = 0,\ A(b, c) = 0,\ A(b, d) = 0,\ A(b, e) = 0.6,\ A(b, f) = 0.6,\ A(b, g) = 0,$
$A(b, h) = 0.6,\ A(b, i) = 0,\ A(b, j) = 0.6,\ A(b, \top) = 0.02,$
$A(c, \bot) = 0,\ A(c, a) = 0,\ A(c, b) = 0,\ A(c, d) = 0,\ A(c, e) = 0,\ A(c, f) = 0.6,\ A(c, g) = 0.6,$
$A(c, h) = 0,\ A(c, i) = 0.6,\ A(c, j) = 0.6,\ A(c, \top) = 0.02,$
$A(d, \bot) = 0,\ A(d, a) = 0,\ A(d, b) = 0,\ A(d, c) = 0,\ A(d, e) = 0,\ A(d, f) = 0,\ A(d, g) = 0,$
$A(d, h) = 0.6,\ A(d, i) = 0.6,\ A(d, j) = 0,\ A(d, \top) = 0.02,$
$A(e, \bot) = 0,\ A(e, a) = 0,\ A(e, b) = 0,\ A(e, c) = 0,\ A(e, d) = 0,\ A(e, f) = 0,\ A(e, g) = 0,$
$A(e, h) = 0.6,\ A(e, i) = 0,\ A(e, j) = 0,\ A(e, \top) = 0.02,$
$A(f, \bot) = 0,\ A(f, a) = 0,\ A(f, b) = 0,\ A(f, c) = 0,\ A(f, d) = 0,\ A(f, e) = 0,\ A(f, g) = 0,$
$A(f, h) = 0,\ A(f, i) = 0,\ A(f, j) = 0.6,\ A(f, \top) = 0.02,$
$A(g, \bot) = 0, A(g, a) = 0,\ A(g, b) = 0,\ A(g, c) = 0,\ A(g, d) = 0,\ A(g, e) = 0,\ A(g, f) = 0,$
$A(g, h) = 0,\ A(g, i) = 0.6,\ A(g, j) = 0.6,\ A(g, \top) = 0.02$
$A(h, \bot) = 0,\ A(h, a) = 0,\ A(h, b) = 0,\ A(h, c) = 0,\ A(h, d) = 0,\ A(h, e) = 0,\ A(h, f) = 0,$
$A(h, g) = 0,\ A(h, i) = 0,\ A(h, j) = 0,\ A(h, \top) = 0.02,$
$A(i, \bot) = 0,\ A(i, a) = 0,\ A(i, b) = 0,\ A(i, c) = 0,\ A(i, d) = 0,\ A(i, e) = 0,\ A(i, f) = 0,$
$A(i, g) = 0,\ A(i, h) = 0,\ A(i, j) = 0,\ A(i, \top) = 0.02,$
$A(j, \bot) = 0,\ A(j, a) = 0,\ A(j, b) = 0,\ A(j, c) = 0,\ A(j, d) = 0,\ A(j, e) = 0,\ A(j, f) = 0,$
$A(j, g) = 0,\ A(j, h) = 0,\ A(j, i) = 0,\ A(j, \top) = 0.04,$
$A(\top, \bot) = 0,\ A(\top, a) = 0,\ A(\top, b) = 0,\ A(\top, c) = 0,\ A(\top, d) = 0,\ A(\top, e) = 0,$
$A(\top, f) = 0,\ A(\top, g) = 0,\ A(\top, h) = 0, A(\top, i) = 0,\ A(\top, j) = 0.$

This fuzzy relation is shown in the following table:

A	$\bot$	a	b	c	d	e	f	g	h	i	j	$\top$
$\bot$	1.0	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
a	0.0	1.0	0.0	0.0	0.6	0.6	0.0	0.0	0.6	0.6	0.0	0.02
b	0.0	0.0	1.0	0.0	0.0	0.6	0.6	0.0	0.6	0.0	0.6	0.02
c	0.0	0.0	0.0	1.0	0.0	0.0	0.6	0.6	0.0	0.6	0.6	0.02
d	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.6	0.6	0.0	0.02
e	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.6	0.0	0.0	0.02
f	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.6	0.02
g	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.6	0.6	0.02
h	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.02
i	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.02
j	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.02
$\top$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\bot$	a	b	c	d	e	f	g	h	i	j	$\top$
$\bot$	$\bot$	a	b	c	d	e	f	g	h	i	j	$\top$
a	a	a	e	i	d	e	$\top$	i	h	i	$\top$	$\top$
b	b	e	b	f	h	e	f	j	h	$\top$	j	$\top$
c	c	i	f	c	i	$\top$	f	g	$\top$	i	j	$\top$
d	d	d	h	i	d	h	$\top$	i	h	i	$\top$	$\top$
e	e	e	e	$\top$	h	e	$\top$	$\top$	h	$\top$	$\top$	$\top$
f	f	$\top$	f	f	$\top$	$\top$	f	j	$\top$	$\top$	j	$\top$
g	g	i	j	g	i	$\top$	j	g	$\top$	i	j	$\top$
h	h	h	h	$\top$	h	h	$\top$	$\top$	h	$\top$	$\top$	$\top$
i	i	i	$\top$	i	i	$\top$	$\top$	i	$\top$	i	$\top$	$\top$
j	j	$\top$	j	j	$\top$	$\top$	j	j	$\top$	$\top$	j	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\bot$	a	b	c	d	e	f	g	h	i	j	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	$\bot$	a	a	$\bot$	$\bot$	a	a	$\bot$	a
b	$\bot$	$\bot$	b	$\bot$	$\bot$	b	b	$\bot$	b	$\bot$	b	b
c	$\bot$	$\bot$	$\bot$	c	$\bot$	$\bot$	c	c	$\bot$	c	c	c
d	$\bot$	a	$\bot$	$\bot$	d	a	$\bot$	$\bot$	d	d	$\bot$	d
e	$\bot$	a	b	$\bot$	a	e	b	$\bot$	e	a	b	e
f	$\bot$	$\bot$	b	c	$\bot$	b	f	c	b	c	f	f
g	$\bot$	$\bot$	$\bot$	c	$\bot$	$\bot$	c	g	$\bot$	g	g	g
h	$\bot$	a	b	$\bot$	d	e	b	$\bot$	h	d	b	h
i	$\bot$	a	$\bot$	c	d	a	c	g	d	i	g	i
j	$\bot$	$\bot$	b	c	$\bot$	b	f	g	b	g	j	j
$\top$	$\bot$	a	b	c	d	e	f	g	h	i	j	$\top$

We note that (X, A) is a fuzzy lattice.
Here $a \wedge_F g = \bot$, $(a \vee_F g) \wedge_F b = i \wedge_F b = \bot$ implies $a \wedge_F (g \vee_F b) = a \wedge_F j = \bot$.
Therefore FGD property is satisfied.
Also, $\bot \prec_F c$ and $a \wedge_F g = \bot$ holds, as $a \vee_F c = i$ so i does not fuzzy cover a.
Hence (X, A) satisfies FGD but not FAC.

Hence, the properties FGD and FAC are stronger conditions than FAD, but they are independent from each other even in fuzzy atomistic lattices.

Proposition 3.2. *A fuzzy modular lattice with a fuzzy least element satisfies FGD.*

Proof. Let $\mathcal{L} = (X, A)$ be a fuzzy modular lattice with fuzzy least element $\bot$.
Let $a, b, c \in X$ with $a \wedge_F b = \bot$ and $(a \vee_F b) \wedge_F c = \bot$. We have to show that the element

$$d = a \wedge_F (b \vee_F c) \tag{5}$$

is $\bot$. By using modularity we get

$$\begin{aligned} b \vee_F c &= (a \vee_F b \vee_F c) \wedge_F (b \vee_F c), \\ &= \{(a \vee_F b) \vee_F c\} \wedge_F (b \vee_F c), \\ &= \{(a \vee_F b) \wedge_F (b \vee_F c)\} \vee_F \{c \wedge_F (b \vee_F c)\}, \\ &= \{(a \vee_F b) \wedge_F (b \vee_F c)\} \vee_F c, \text{ by absorption} \\ &= e \vee_F c \end{aligned}$$

i.e.,

$$b \vee_F c = e \vee_F c \tag{6}$$

where $e = (a \vee_F b) \wedge_F (b \vee_F c)$. On the other hand

$$\begin{aligned} e \wedge_F c &= (a \vee_F b) \wedge_F (b \vee_F c) \wedge_F c, \\ &= (a \vee_F b) \wedge_F c, \text{ by absorption} \\ e \wedge_F c &= \bot, \quad \text{as } (a \vee_F b) \wedge_F c = \bot \end{aligned}$$

i.e.,

$$e \wedge_F c = \bot. \tag{7}$$

Hence again by modularity

$$\begin{aligned} e &= e \wedge_F (e \vee_F c), \\ &= e \wedge_F (b \vee_F c), \text{ by (6)} \\ &= b \vee_F (e \wedge_F c), \text{ by modularity i.e., } e \wedge_F (c \vee_F b) = (e \wedge_F c) \vee_F b \\ &= b \vee_F \bot, \text{ by (7)} \\ &= b \end{aligned}$$

By (5) we get

$$A(d,a) > 0. \tag{8}$$

As $A(a, a \wedge_F e) > 0$ and $b = e$ we get

$$A(a, a \wedge_F b) > 0. \tag{9}$$

From (8) and (9) by fuzzy transitivity we get $A(d, a \wedge_F b) > 0$. But $a \wedge_F b = \perp$. Therefore we get

$$A(d, \perp) > 0. \tag{10}$$

As

$$A(\perp, d) > 0 \tag{11}$$

always holds.
Therefore from (10) and (11) we get $d = \perp$.
Putting $d = \perp$ in (5) we get $a \wedge_F (b \vee_F c) = \perp$. □

Definition 3.6. *A fuzzy lattice (X, A) is called fuzzy sectionally complemented lattice(FSC) if it satisfies the following condition:*
If for $a \in X$ with $0 < A(a, \top) < 1$, then there exists a fuzzy atom $p \in X$ such that $a \wedge_F p = \perp$.

Proposition 3.3. *A fuzzy sectionally complemented lattice satisfying FGD is fuzzy $\perp$-modular pair.*

Proof. Let (X, A) be a FSC lattice satisfying FGD. Let $a, b, c \in X$.
To show that (a, b) is a fuzzy $\perp$-modular pair, that is $(c \vee_F a) \wedge_F b = c \vee_F (a \wedge_F b)$ for every $A(c, b) > 0$.
Case i) Assume that $a \wedge_F b = \perp$.
If $A(c, b) > 0$, then $A(c \vee_F (a \wedge_F b), (c \vee_F a) \wedge_F b) > 0$.
As $a \wedge_F b = \perp$ so we have $A(c, (c \vee_F a) \wedge_F b) > 0$.
By the assumption there exists an element $c^{'} \in X$ with $c \wedge_F c^{'} = \perp$ and $c \vee_F c^{'} = (c \vee_F a) \wedge_F b$.
Taking meet a on both sides we get

$$\begin{aligned}(c \vee_F c^{'}) \wedge_F a &= \{(c \vee_F a) \wedge_F b\} \wedge_F a,\\ &= (c \vee_F a) \wedge_F b \wedge_F a,\\ &= (c \vee_F a) \wedge_F \perp, \quad \text{as } a \wedge_F b = \perp\\ &= \perp\end{aligned}$$

i.e., $(c \vee_F c^{'}) \wedge_F a = \perp$.
Applying the property to the triplet $(c^{'}, c, a)$ we get $c^{'} \wedge_F (c \vee_F a) = \perp$. As

$$A(c^{'}, (c \vee_F c^{'}) \wedge_F a) > 0 \tag{12}$$

and

$$A((c \vee_F c^{'}) \wedge_F a, c \wedge_F c^{'}) > 0. \tag{13}$$

But $c \vee_F c^{'} = (c \vee_F a) \wedge_F b$. Therefore (13) reduces to

$$A((c \vee_F c^{'}) \wedge a, (c \vee_F a) \wedge_F b) > 0. \tag{14}$$

From (12) and (14) by fuzzy transitivity of A we get $A(c^{'}, (c \vee_F a) \wedge_F b) > 0$.
But $(c \vee_F a) \wedge_F b = \perp$. Therefore

$$A(c^{'}, \perp) > 0. \tag{15}$$

As

$$A(\perp, c^{'}) > 0 \tag{16}$$

always holds.
Therefore form (15) and (16) we get $c^{'} = \bot$. Hence $c = (c \vee_F a) \wedge_F b$ and (a, b) is a fuzzy $\bot$-modular pair.
Case ii) Let $a, b \in X$ and $A(c, b) > 0$. Let the element $d \in X$ with $d = a \wedge_F b$ has a complement $d^{'} \in X$ such that $A(\bot, d^{'}) > 0$ and $A(d^{'}, a) > 0$. Since

$$\begin{aligned} d^{'} \wedge_F b &= d^{'} \wedge_F a \wedge_F b, \\ &= d^{'} \wedge_F d, \\ &= \bot \end{aligned}$$

This shows that $(d^{'}, b)$ is fuzzy $\bot$-modular pair.
Using this and the observations $A(c \vee_F d, b) > 0$ we get

$$\begin{aligned} (c \vee_F a) \wedge_F b &= (c \vee_F d \vee_F d^{'}) \wedge_F b, \\ &= (c \vee_F d) \vee_F (d^{'} \wedge_F b), \quad (d^{'}, b) \text{ is fuzzy } \bot\text{-modular pair.} \\ &= (c \vee_F d) \vee_F \bot, \\ &= c \vee_F d, \\ &= c \vee_F (a \wedge_F b), \quad \text{ as } d = a \wedge_F b \end{aligned}$$

Hence (a, b) is a fuzzy $\bot$-modular pair. □

Khubchandani and Khubchandani [4] defined fuzzy $\bot$-distributive lattices in fuzzy lattices. Using the same idea of FGD we define fuzzy $\bot$-general disjointness property ($\bot$-FGD), fuzzy $\bot$-atomic disjointness ($\bot$-FAD) and fuzzy $\bot$-atomic exchange ($\bot$-FAE).

Definition 3.7. *A fuzzy lattice (X, A) with $\bot$ is called fuzzy $\bot$-general disjointness property ($\bot$-FGD) if $a \wedge_F b = \bot$ and $(a \vee_F b) \wedge_F c = \bot$ imply $a \wedge_F (b \vee_F c) = \bot$, for all $a, b, c \in X$.*

Definition 3.8. *A fuzzy lattice (X, A) with $\bot$ is called fuzzy $\bot$-atomic disjointness ($\bot$-FAD) if for a fuzzy atom $p \in X$, $\bot \prec_F p$, $a \wedge_F b = \bot$ and $(a \vee_F b) \wedge_F p = \bot$ imply $a \wedge_F (b \vee_F p) = \bot$, for all $a, b \in X$.*

Definition 3.9. *A fuzzy lattice (X, A) with $\bot$ is called fuzzy $\bot$-atomic exchange ($\bot$-FAE) if for fuzzy atoms $p, q \in X$, $\bot \prec_F p$, $\bot \prec_F q$, $a \wedge_F p = \bot$ and $A(p, a \vee_F q) > 0$ imply $A(q, a \vee_F p) > 0$ for $a \in X$.*

Proposition 3.4. *Let (X, A) be a fuzzy lattice with $\bot$. If (X, A) is FAD then (X, A) is FAE. If (X, A) is fuzzy atomic lattice then fuzzy $\bot$-FAD and fuzzy $\bot$-FAE are equivalent conditions.*

Proof. Proof follows along the same lines as the proof of Proposition 3.1. □

Theorem 3.2. *If a fuzzy complemented lattice is both fuzzy $\bot$-distributive and fuzzy $\bot$-modular, then it is fuzzy distributive.*

Proof. Let (X, A) be a fuzzy complemented lattice which is both fuzzy $\bot$-distributive and fuzzy $\bot$-modular.
We first show that (X, A) is fuzzy modular.
Let $a, b, c \in X$ with $A(c, b) > 0$ and let $d = (a \wedge_F b) \vee_F c$ and $e = (a \vee_F c) \wedge_F b$. We have to show that $d = e$. Let $d^{'}$ be fuzzy complement of d. The element $b \wedge_F d^{'}$ is disjoint from a and d, hence also from

$$a \vee_F d = a \vee_F c \tag{17}$$

by fuzzy $\perp$-distributivity.

$$\begin{aligned} d^{'} \wedge_F e &= d^{'} \wedge_F (a \vee_F c) \wedge_F b, \\ &= (b \wedge_F d^{'}) \wedge_F (a \vee_F c), \\ &= (b \wedge_F d^{'}) \wedge_F (a \vee_F d), \quad \text{from (17)} \\ &= \perp \end{aligned}$$

The fuzzy $\perp$-modularity of (X, A) implies now that the pair $(d^{'}, e)$ is $\perp$-modular and hence

$$d = (d \vee_F d^{'}) \wedge_F e = \top \wedge_F e = e.$$

Hence (X, A) is fuzzy modular.
If (X, A) is not fuzzy distributive. Let $a, b, c, d, e \in X$ with

$$d = a \wedge_F b = a \wedge_F c = b \wedge_F c$$

and

$$e = a \vee_F b = a \vee_F c = b \vee_F c.$$

Let $c^{'}$ be a fuzzy complement of c. The element $b \wedge c^{'}$ is then disjoint from a and c, hence also from $a \vee_F c = e$. This means that $b \wedge_F c^{'} = \perp$. Further, $e \wedge_F c^{'} = \perp$, by fuzzy $\perp$-distributivity again and thus $c = e$ by fuzzy $\perp$-modularity. This is however a contradiction and thus (X, A) is distributive. □

4. Fuzzy Disjointness Properties in Fuzzy Semi-complemented Lattices

The notion of fuzzy Birkhoff lattice, introduced by Wasadikar and Khubchandani [24], has motivated us to introduce and study fuzzy disjointness in Fuzzy Semi-complemented lattices. In this section, we will prove that a fuzzy complemented $\perp$-modular lattice is fuzzy Birkhoff lattice.

Theorem 4.1. *Let (X, A) be a fuzzy section semicomplemented lattice(FSSC). If (X, A) has the fuzzy atomic disjointness property, then it has fuzzy atomic covering property(FAC). If (X, A) is fuzzy atomic, then it is fuzzy Birkhoff lattice.*

Proof. Let $p \in X$ be fuzzy atom with $\perp \prec_F p$ and $a \wedge_F p = \perp$.
We have to prove that $a \prec_F a \vee_F p$. Suppose there is an element $b \in X$ such that $0 < A(a, b) < 1$ and $0 < A(b, a \vee_F p) < 1$. As (X, A) is FSSC, there is a non-zero element $c \in X$ such that $A(c, b) > 0$ with $c \wedge_F a = \perp$. By Corollary 2.1 we get $A(a \vee_F c, b) > 0$. Taking meet p on both sides we get

$$A((a \vee_F c) \wedge_F p, b \wedge_F p) > 0. \tag{18}$$

But $b \wedge_F p = \perp$. Therefore (18) reduces to

$$A((a \vee_F c) \wedge_F p, \perp) > 0. \tag{19}$$

As

$$A(\perp, (a \vee_F c) \wedge_F p) > 0 \tag{20}$$

always holds.
Therefore from (19) and (20) by fuzzy antisymmetry we get $(a \vee_F c) \wedge_F p = \perp$.
By using fuzzy atomic disjointness property we obtain

$$c \wedge_F (a \vee_F p) = \perp \quad or \quad c = \perp,$$

Which is a contradiction. Hence (X, A) satisfies the fuzzy atomic covering property.
Let now (X, A) be fuzzy atomic and let $a \wedge_F b \prec_F b$. As fuzzy section semi-complemented and fuzzy atomic lattices are fuzzy atomistic, there is fuzzy atom $p \in X$ such that

$A(p, b) > 0$ with $(a \wedge_F b) \wedge_F p = \perp$. Since b covers $a \wedge b$, it follows that $b = (a \wedge_F b) \vee_F p$ and since (X, A) has fuzzy atomic covering property, we obtain $a \prec_F a \vee_F p$ which implies $a \vee_F p = a \vee_F (a \wedge_F b) \vee_F p = a \vee_F b$. This proves that (X, A) is fuzzy Birkhoff lattice. $\square$

Corollary 4.1. *A fuzzy atomistic lattice (X, A) with the fuzzy exchange property is fuzzy Birkhoff lattice.*

Proof. A fuzzy atomistic lattice is both fuzzy atomistic and fuzzy section semi-complemented. Moreover by Propostion 3.1, a fuzzy atomistic lattice has the fuzzy atomic disjointness property whenever it has the atomic exchange property. $\square$

We define fuzzy strong neat using the terminology of Stern [19].

Definition 4.1. *A fuzzy lattice (X, A) is strong neat if for every pair (a, b) with $a \prec_F b$ there is a fuzzy atom $p \in X$ such that $a \vee_F p = b$.*

Theorem 4.2. *Let (X, A) be a fuzzy lattice. If a fuzzy strongly neat lattice has the fuzzy atomistic covering property, then it is fuzzy Birkhoff lattice.*

Proof. Let (X, A) be a fuzzy lattice. In a fuzzy strongly neat lattice (X, A) with the fuzzy atomic covering property. Let $a \wedge_F b \prec_F b$. Since (X, A) is fuzzy strongly neat, there is an fuzzy atom $p \in X$ such that $(a \wedge_F b) \vee_F p = b$. Now $a \wedge_F p = \perp$ and hence $a \prec_F a \vee_F p$, by the fuzzy atomic covering property. Since $a \vee_F p = a \vee_F (a \wedge_F b) \vee_F p = a \vee_F b$, we have proved that (X, A) fuzzy Birkhoff lattice. $\square$

Theorem 4.3. *Let (X, A) be a fuzzy lattice. If fuzzy section semi-complemented lattice has the disjointness property, then it is fuzzy $\perp$-modular and fuzzy Birkhoff lattice.*

Proof. Let (X, A) be a fuzzy section semicomplemented lattice with fuzzy disjointess property. Let $a \wedge_F b = \perp$ and $A(c, b) > 0$ for $a, b, c \in X$.
Similarly as in the first part of Theorem 4.1 it can be shown that the element $c \in X$ cannot have any non-zero fuzzy complement in $(a \vee_F c) \wedge_F b$.
Thus $c = (a \vee_F c) \wedge_F b$, and disjoint pair (a, b) is fuzzy modular. This shows that (X, A) is fuzzy $\perp$-modular.
To show that (X, A) is fuzzy Birkhoff lattice.
Let $a \wedge b \prec_F b$ and $d = a \wedge_F b$. As (X, A) is fuzzy section semicomplemented, there is a non-zero element $c \in X$ such that $d \wedge_F c = \perp$ and $A(c, b) > 0$.
Since b fuzzy covers d it follows that $b = d \vee_F c$. By fuzzy $\perp$-modularity (c, d) is a modular pair and thus by Lemma 2.1, the element c is a fuzzy atom. Since now $a \vee_F b = a \vee_F c$, the assertion $a \prec_F a \vee_F b$ follows from the fuzzy atomic covering property implied by the fuzzy $\perp$-modularity. $\square$

Now we shall prove that a fuzzy complemented $\perp$-modular lattice is fuzzy Birkhoff lattice.

Corollary 4.2. *A fuzzy complemented $\perp$-modular lattice is fuzzy Birkhoff lattice.*

Proof. It is sufficient to note that a complemented $\perp$-modular lattice (X, A) is fuzzy section semicomplemented. If $0 < A(a, b) < 1$ and a' is a complement of a, then by fuzzy $\perp$-modularity, the element $b \wedge_F a'$ is non-zero and is thus fuzzy semicomplement of a in b. $\square$

5. Conclusions

In this paper, we have presented the concept of the fuzzy general disjointness property, along with its corresponding notation. We also defined several related concepts, including fuzzy atomic disjointness property, fuzzy atomic exchange property, and fuzzy atomic covering property, and explored the relationships among them. Furthermore, we demonstrated that if a fuzzy section semi-complemented lattice possesses the fuzzy disjointness property, it will necessarily be both fuzzy $\perp$-modular and a fuzzy Birkhoff lattice.

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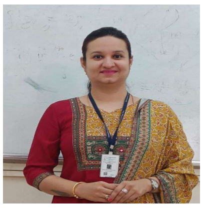

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