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A CHARACTERIZATION OF E-COMPLETENESS IN VECTOR METRIC SPACES WITH AN APPLICATION IN FIXED POINT THEORY

SUSHANTA KUMAR MOHANTA[1,*], SHUBHA DAS[2], §

Abstract. The main purpose of this article is to introduce the notion of d_v-point in a vector metric space which is a generalization of the notion of d-point in metric spaces and extend Weston's characterization of metric completeness to vector metric spaces in terms of d_v-point. In fact, we have utilized the concepts of lower semicontinuity and uniform continuity in this new framework to establish the main result. Finally, we established relations among minimal points, d_v-points and fixed points in this new setting. As an application of this study, we obtained the analogue of Banach Contraction Principle in vector metric spaces.

Keywords: d_v-point, E-completeness, minimal point, fixed point.

AMS Subject Classification: 54H25, 47H10.

1. Introduction

Metric completeness is closely related to metric fixed point theory. Several authors successfully characterized metric completeness in terms of fixed point theory (see [11, 12, 13, 14, 16, 18, 19]). There are a lot of generalizations of the concept of metric spaces such as b-metric space, introduced by Bakhtin [3], partial metric space by Matthews [9], and dislocated metric space by Hitzler et al. [8]. Recently, Xu et al. [21] introduced a kind of new convergence for sequences and a new kind of completeness in cone b-metric spaces over Banach algebras and established a common fixed point theorem for such spaces. After wards, Y. Han and others [7], introduced several b-generalized contractive mappings in cone b-metric spaces over Banach algebras and obtained some important fixed point theorems for asymptotically regular mappings by weakening the completeness of the spaces. In 2009, Çevic et al. [5] introduced the concept of vector metric spaces as a generalization of metric spaces, where the metric is Riesz space valued and studied some properties

[1] Department of Mathematics, West Bengal State University, Barasat, 24 Parganas (North), Kolkata-700126, West Bengal, India.
e-mail: smwbes@yahoo.in; ORCID: https://orcid.org/0000-0002-8603-5380.
[2] Department of Mathematics, West Bengal State University, Barasat, 24 Parganas (North), Kolkata-700126, West Bengal, India.
e-mail: iamshubhadas; ORCID: https://orcid.org/0009-0000-4382-4294.
[*] Corresponding author
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of such spaces. In [17], the authors discussed Reich-Perov-type contractive mappings using a novel concept in vector valued metric spaces and gave sufficient conditions for the existence of fixed points for such contractive mappings. Very recently, Çevik et al. [6] proved some coupled fixed point theorems for the functions having mixed monotone properties on ordered vector metric spaces. In 1977, J. D. Weston [20] had characterized metric completeness by means of d-point for lower semicontinuous functions. In this study, our main purpose is to introduce the concept of d_v-point in vector metric spaces and extend Weston's characterization [20] in such spaces in terms of d_v-point. Finally, we apply this new characterization to obtain some important fixed point results in E-complete vector metric spaces.

2. Some Basic Concepts

In this section, we recall some basic facts about Riesz spaces mostly of which can be found in [2, 5].

A partially ordered set $(X, \preceq)$ is called a lattice if each pair of elements $x, y \in X$ has a supremum and an infimum.
A real vector space E with an order relation $\preceq$ on E that is compatible with the algebraic structure of E in the sense that satisfies properties:

(1) $x \preceq y$ implies $x + z \preceq y + z$ for each $z \in E,\ x, y \in E$;
(2) $x \preceq y$ implies $tx \preceq ty$ for each $t > 0$, $x, y \in E$

is called an ordered vector space.
An ordered vector space that is also a lattice is called a Riesz space or vector lattice. Throughout this paper, we take θ as the zero vector of the vector space E. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$. Let E be a Riesz space with the positive cone $E_+ = \{x \in E : \theta \preceq x\}$. If (a_n) is a decreasing sequence in E such that $\inf a_n = a$, we write $a_n \downarrow a$. Similarly, if (a_n) is an increasing sequence in E such that $\sup a_n = a$, we write $a_n \uparrow a$.

Some basic properties of decreasing sequences can be stated as follows:

(i) $a_n \downarrow a$ and $b_n \downarrow b \Rightarrow a_n + b_n \downarrow a + b$;
(ii) $a_n \downarrow a \Rightarrow \lambda a_n \downarrow \lambda a$ for $\lambda > 0$ and $\lambda a_n \uparrow \lambda a$ for $\lambda < 0$;
(iii) $a_n \downarrow a$ and $b_n \downarrow b \Rightarrow a_n \vee b_n \downarrow a \vee b$ and $a_n \wedge b_n \downarrow a \wedge b$, where $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$.

The Riesz space E is said to be Archimedean if $\frac{1}{n}a \downarrow \theta$ holds for every $a \in E_+$.

A Riesz space E is called order complete or Dedekind complete if every nonempty subset of E which is bounded from above (below) has a supremum (infimum). Any order complete Riesz space is Archimedean but the converse is not true, in general.

A sequence (b_n) in E is called order convergent or o-convergent to b if there exists a sequence (a_n) in E satisfying $a_n \downarrow \theta$ and $| b_n - b | \preceq a_n$ for all n, and written $b_n \xrightarrow{o} b$ or $o - \lim b_n = b$, where $| a | = \sup\{a, -a\}$ for any $a \in E$. Moreover, (b_n) is called order-Cauchy or o-Cauchy if there exists a sequence (a_n) in E such that $a_n \downarrow \theta$ and $| b_n - b_{n+p} | \preceq a_n$ holds for all n and p. E is called o-complete if every o-Cauchy sequence is o-convergent. For other notations and facts about Riesz spaces, we refer to [1].

Lemma 2.1. [2] *If E is a Riesz space and $a \preceq ka$ where $a \in E_+$ and $k \in [0, 1)$, then $a = \theta$.*

Definition 2.1. *A Riesz space E is called regular if every decreasing sequence in E which is bounded from below is o-convergent.*

Remark 2.1. *If E is Dedekind complete and totally ordered, then as a consequence it follows that E is regular.*

Definition 2.2. [5] *Let X be a nonempty set and E be a Riesz space. The function $d_v : X \times X \to E$ is said to be a vector metric (or E-metric) if it satisfies the following properties:*

(v_1) $d_v(x, y) = \theta$ *if and only if* $x = y$;

(v_2) $d_v(x, y) \preceq d_v(x, z) + d_v(y, z)$ *for all* $x, y, z \in X$.

The triple (X, d_v, E) is said to be a vector metric space.

It is easy to observe that vector metric spaces generalize metric spaces.

In a vector metric space (X, d_v, E), the following assertions hold:

(i) $\theta \preceq d_v(x, y)$ for all $x, y \in X$;

(ii) $d_v(x, y) = d_v(y, x)$ for all $x, y \in X$;

(iii) $| d_v(x, z) - d_v(y, z) | \preceq d_v(x, y)$ for all $x, y, z \in X$;

(iv) $| d_v(x, z) - d_v(y, w) | \preceq d_v(x, y) + d_v(z, w)$ for all $x, y, z, w \in X$.

Example 2.1. [5] *A Riesz space E is a vector metric space with $d_v : E \times E \to E$ defined by*

$$d_v(x, y) = | x - y | .$$

This vector metric is said to be absolute valued metric on E.

Example 2.2. [5] *It is well known that $\mathbb{R}^2$ is a Riesz space with coordinate wise ordering defined by*

$$(x_1, y_1) \preceq (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2$$

for (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$.

Again, $\mathbb{R}^2$ is a Riesz space with lexicographical ordering defined by

$$(x_1, y_1) \preceq (x_2, y_2) \text{ if and only if } x_1 < x_2 \text{ or } x_1 = x_2,\ y_1 \leq y_2.$$

It is worth noting that $\mathbb{R}^2$ is Archimedean with coordinate wise ordering but not with lexicographical ordering.

Then, $d_v : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$d_v((x_1, y_1), (x_2, y_2)) = (\alpha | x_1 - x_2 |, \beta | y_1 - y_2 |)$$

is a vector metric, where α, β are positive real numbers.

Example 2.3. [5] *Let $d_v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be defined by*

$$d_v(x, y) = (\alpha | x - y |, \beta | x - y |)$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$. Then d_v is a vector metric with coordinate wise or lexicographical ordering.

Example 2.4. *Let $C[a, b]$ be the space of all real valued continuous functions defined on $[a, b]$. For $f, g \in C[a, b]$, we define $f \preceq g$ if and only if $f(x) \leq g(x)$ for all $x \in [a, b]$. Then $C[a, b]$ is a Riesz space. For a fixed $f \in C[a, b]$, where $f(x) > 0$ for all x, we define $d_v : \mathbb{R} \times \mathbb{R} \to C[a, b]$ by*

$$d_v(x, y) = | x - y | f.$$

Then d_v is a vector metric.

Definition 2.3. [5] *Let (X, d_v, E) be a vector metric space and let (x_n) be a sequence in X. Then*

(i) *(x_n) vectorial converges or E-converges to a point $x \in X$, written $x_n \xrightarrow{d_v,E} x$, if there is a sequence (a_n) in E satisfying $a_n \downarrow \theta$ and $d_v(x_n, x) \preceq a_n$ for all n.*

(ii) *(x_n) is called vectorially Cauchy or E-Cauchy, if there is a sequence (a_n) in E such that $a_n \downarrow \theta$ and $d_v(x_n, x_{n+p}) \preceq a_n$ holds for all n and p.*

(iii) *(X, d_v, E) is said to be E-complete if each E-Cauchy sequence in X E-converges to a point in X.*

(iv) *A subset Y in (X, d_v, E) is said to be E-closed whenever $(x_n) \subseteq Y$ and $x_n \xrightarrow{d_v,E} x$ imply $x \in Y$.*

Remark 2.2. *It is easy to observe that there is a relationship between E-convergence on X with the o-convergence on E. In fact, $x_n \xrightarrow{d_v,E} x$ if and only if $d(x_n, x) \xrightarrow{o} \theta$.*

Remark 2.3. *If $E = \mathbb{R}$, then the notion of E-convergence coincides with the notion of metric convergence. Similarly, the concept of E-Cauchy sequence coincides with the concept of metric Cauchy sequence.*

Remark 2.4. *If $X = E$ and d_v is the absolute valued vector metric on X, then the concept of vectorially convergence coincides with the concept of order convergence.*

Lemma 2.2. [5] *Let (X, d_v, E) be a vector metric space and let $x_n \xrightarrow{d_v,E} x$. Then, we have the following properties:*

(i) *The limit x is unique.*

(ii) *Every subsequence of (x_n) E-converges to x.*

(iii) *If also $y_n \xrightarrow{d_v,E} y$, then $d_v(x_n, y_n) \xrightarrow{o} d_v(x, y)$.*

Definition 2.4. *Let E be Dedekind complete and let (X, d_v, E) be a vector metric space. A function $\varphi : X \to E$ which is bounded from below is called a lower semicontinuous function on X whenever*

$$x_n \xrightarrow{d_v,E} x \Longrightarrow \varphi(x) \preceq \lim_{n\to\infty} \inf \varphi(x_n) := \sup_{n\geq 1} \inf_{m\geq n} \varphi(x_m).$$

Definition 2.5. *Let (X, d_v, E) be a vector metric space and $f : X \to E$. Then, the function f is called uniformly continuous on X if for any $\epsilon \succ \theta$ there is a $c \in E$ with $c \succ \theta$ such that*

$$d_v(x, y) \prec c \Longrightarrow \mid f(x) - f(y) \mid \prec \epsilon.$$

3. Main Results

We begin with a definition.

Definition 3.1. *Let (X, d_v, E) be a vector metric space and $h : X \to E$. A point $x_0 \in X$ is called a d_v-point for h if for every other point $x \in X$ with $h(x_0) - h(x) \succ \theta$,*

$$h(x_0) - h(x) \prec d_v(x_0, x).$$

Example 3.1. *Let $E=\mathbb{R}^2$, $X=[0,\infty)$ and $d_v : X\times X \to E$ be defined by*

$$d_v(x,y)=(\mid x-y\mid, \mid x-y\mid).$$

Then (X,d_v,E) is a vector metric space with coordinate wise ordering.
We define $h : X\to E$ by

$$h(x)=\left(-\frac{x}{2},-\frac{x}{2}\right) \text{ for all } x\in X.$$

Then,

$$h(0)-h(x)=(0,0)-\left(-\frac{x}{2},-\frac{x}{2}\right)=\left(\frac{x}{2},\frac{x}{2}\right)\prec d_v(0,x)$$

for every $x\in X$ with $x\neq 0$. Also, $h(0)-h(x)\succ\theta$. Thus, 0 is a d_v-point for h.

Theorem 3.1. *Let E be a Riesz space which is Dedekind complete and totally ordered. Let (X,d_v,E) be an E-complete vector metric space. Then any lower semicontinuous function $h : X\to E$ has a d_v-point. If (X,d_v,E) is not E-complete but E is o-complete and E_+ is closed w.r.t. absolute valued vector metric on E, then there is a uniformly continuous function $g : X\to E$ which is bounded below but has no d_v-point.*

Proof. For any point $x_1\in X$, we construct a sequence (x_n) in the following way:
For each $n\in\mathbb{N}$, let

$$\alpha(x_n)=\inf\{h(x) : h(x_n)-h(x)\succeq d_v(x_n,x)\succ\theta\}=\inf A,$$

where $A=\{h(x) : h(x_n)-h(x)\succeq d_v(x_n,x)\succ\theta\}$.
Since h is bounded below and E is Dedekind complete, it follows that $\alpha(x_n)$ exists.

Let x_{n+1} be a point in X such that

$$h(x_n)-h(x_{n+1})\succeq d_v(x_n,x_{n+1})\succ\theta \tag{1}$$

and

$$h(x_{n+1})\prec\alpha(x_n)+\frac{c_0}{n}, \tag{2}$$

where $c_0\succ\theta$ is a fixed element of E.

In fact, if $\alpha(x_n)+\frac{c_0}{n}\preceq h(x)$ for all $h(x)\in A$, and this is possible since E is totally ordered, then $\alpha(x_n)+\frac{c_0}{n}$ becomes a lower bound of A satisfying $\alpha(x_n)\prec\alpha(x_n)+\frac{c_0}{n}$ which contradicts the fact that $\alpha(x_n)=\inf A$. The preceding discussion ensures the existence of $x_{n+1}\in X$ satisfying conditions (1) and (2).

It is worthy to note that in above construction, we have considered none of x_n as a d_v-point for h. Because, if x_n is a d_v point for h, then we have nothing to prove.

Condition (1) guarantees that the sequence $(h(x_n))$ is nonincreasing in E. As h is bounded below, it follows that the sequence $(h(x_n))$ is also bounded below. Since E is Dedekind complete and totally ordered, it is also regular. So, we obtain that the sequence $(h(x_n))$ is o-convergent and hence it is o-Cauchy. So, there exists a sequence (a_n) in E such that $a_n\downarrow\theta$ and $\mid h(x_n)-h(x_{n+p})\mid\preceq a_n$ holds for all n and p.

For $m \geq n$, we have

$$\begin{aligned} h(x_n) - h(x_m) &= h(x_n) - h(x_{n+1}) + h(x_{n+1}) - h(x_{n+2}) \\ &\quad + \cdots + h(x_{m-1}) - h(x_m) \\ &\succeq d_v(x_n, x_{n+1}) + d_v(x_{n+1}, x_{n+2}) + \cdots + d_v(x_{m-1}, x_m) \\ &\succeq d_v(x_n, x_m). \end{aligned} \tag{3}$$

Hence,

$$d_v(x_n, x_m) \preceq h(x_n) - h(x_m) \preceq | h(x_n) - h(x_m) | .$$

Taking $p = m - n$, we get

$$d_v(x_n, x_{n+p}) \preceq | h(x_n) - h(x_{n+p}) | \preceq a_n, \ \forall \ n, \ p.$$

This proves that the sequence (x_n) is E-Cauchy in X. The E-completeness of X implies that the sequence (x_n) is E-convergent to some point in X, say x_0, that is, $x_n \xrightarrow{d_v, E} x_0$.

From (3), it follows that

$$h(x_m) \preceq h(x_n) - d_v(x_n, x_m) \tag{4}$$

for all $m \geq n$. In view of condition (4), Lemma 2.2 and lower semicontinuity of the function h, one can compute that

$$\begin{aligned} h(x_0) &\preceq \lim_{m\to\infty} \inf \, h(x_m) \\ &\preceq \lim_{m\to\infty} \inf \, [h(x_n) - d_v(x_n, x_m)] \\ &= h(x_n) - d_v(x_n, x_0) \end{aligned}$$

for all $n \geq 1$. Thus,

$$h(x_n) - h(x_0) \succeq d_v(x_n, x_0) \tag{5}$$

for all $n \geq 1$.

If x_0 is not a d_v-point for h, then for some $x (\neq x_0) \in X$,

$$h(x_0) - h(x) \succeq d_v(x_0, x) \succ \theta. \tag{6}$$

Using (5) and (2), we obtain

$$h(x) \preceq h(x_{n+1}) + h(x) - h(x_0) \prec \alpha(x_n) + \frac{c_0}{n} + h(x) - h(x_0). \tag{7}$$

Since $c_0 \succ \theta$ and $h(x_0) - h(x) \succ \theta$, we claim that there exists $n \in \mathbb{N}$ such that

$$\frac{c_0}{n} \prec h(x_0) - h(x).$$

In fact, if $h(x_0) - h(x) \preceq \frac{c_0}{n}$ for all $n \in \mathbb{N}$, then $h(x_0) - h(x)$ becomes a lower bound the sequence $(\frac{c_0}{n})$. Therefore, $h(x_0) - h(x) \preceq \inf \dfrac{c_0}{n}$. Since E is Dedekind complete, it is also Archimedean and so $\frac{c_0}{n} \downarrow \theta$. Thus, we obtain that $h(x_0) - h(x) \preceq \theta$, a contradiction and our claim is justified.

Our preceding discussion implies that $\alpha(x_n) + \dfrac{c_0}{n} + h(x) - h(x_0) \prec \alpha(x_n)$. Thus, condition (7) assures that $h(x) \prec \alpha(x_n)$.

From conditions (5) and (6), we obtain that

$$\begin{aligned} h(x_n) - h(x) &= h(x_n) - h(x_0) + h(x_0) - h(x) \\ &\succeq d_v(x_n, x_0) + d_v(x_0, x) \\ &\succ \theta \end{aligned}$$

which implies that $h(x_n) \succ h(x)$. So, $x_n \neq x$ and therefore $d_v(x_n, x) \succ \theta$.
Moreover,

$$h(x_n) - h(x) \succeq d_v(x_n, x_0) + d_v(x_0, x) \succeq d_v(x_n, x) \succ \theta.$$

The definition of $\alpha(x_n)$ ensures that $h(x) \succeq \alpha(x_n)$ which contradicts the fact that $h(x) \prec \alpha(x_n)$. Thus, x_0 is a d_v-point for h.

For the last part, we suppose that (X, d_v, E) is not E-complete but E is o-complete. Then there exists an E-Cauchy sequence (x_n) in X which is not E-convergent. We now compute that for any $x \in X$, the sequence $(2d_v(x, x_n))$ is o-Cauchy in E.
For $x \in X$, we have

$$d_v(x, x_n) \preceq d_v(x, x_m) + d_v(x_m, x_n)$$

which implies that

$$d_v(x, x_n) - d_v(x, x_m) \preceq d_v(x_m, x_n). \tag{8}$$

Interchanging n and m, we obtain

$$d_v(x, x_m) - d_v(x, x_n) \preceq d_v(x_m, x_n). \tag{9}$$

Conditions (8) and (9) together imply that

$$\mid d_v(x, x_n) - d_v(x, x_m) \mid \preceq d_v(x_m, x_n) \text{ for all } m,\, n \in \mathbb{N}. \tag{10}$$

Suppose that $m > n$ and we put $p = m - n$. Then condition (10) reduces to

$$\mid d_v(x, x_n) - d_v(x, x_{n+p}) \mid \preceq d_v(x_n, x_{n+p}) \text{ for all } n,\, p. \tag{11}$$

As (x_n) is E-Cauchy, it follows from (11) that the sequence $(2d_v(x, x_n))$ is o-Cauchy in E and hence it is o-convergent, E being o-complete. Let $g(x) \in E$ be its limit. Since $2d_v(x, x_n) \in E_+$ and E_+ being closed w.r.t. absolute valued vector metric on E, it follows that $g(x) \in E_+$. Clearly, $g(x) \succ \theta$. In fact, if $g(x) = \theta$, then there is a sequence (a_n) in E satisfying $a_n \downarrow \theta$ and $\mid 2d_v(x_n, x) - \theta \mid \preceq a_n$ for all n. This gives that,

$$2d_v(x_n, x) \preceq \mid 2d_v(x_n, x) \mid \preceq a_n \text{ for all } n.$$

Thus, the sequence (x_n) becomes E-convergent which contradicts our assumed hypothesis. Therefore, we obtained that the function g is bounded below.

If $x_0 \in X$, then $g(x_0) = o - \lim 2d_v(x_0, x_n)$. Also, $g(x) = o - \lim 2d_v(x, x_n)$. Then, there are sequences $(a_n),\ (b_n)$ in E satisfying $a_n \downarrow \theta,\ b_n \downarrow \theta$ such that

$$\mid 2d_v(x_n, x_0) - g(x_0) \mid \preceq a_n \text{ and } \mid 2d_v(x_n, x) - g(x) \mid \preceq b_n \text{ for all } n. \tag{12}$$

Now,

$$\begin{aligned} \mid 2d_v(x_n, x_0) - 2d_v(x_n, x) - (g(x_0) - g(x)) \mid \quad &\preceq \quad \mid 2d_v(x_n, x_0) - g(x_0) \mid \\ &\quad + \mid 2d_v(x_n, x) - g(x) \mid \\ &\preceq \quad (a_n + b_n) \text{ for all } n, \end{aligned}$$

where $(a_n + b_n) \downarrow \theta$. This proves that

$$g(x_0) - g(x) = o - \lim[2d_v(x_n, x_0) - 2d_v(x_n, x)] \preceq 2d_v(x_0, x). \tag{13}$$

Interchanging x_0 and x, we obtain

$$g(x) - g(x_0) \preceq 2d(x_0, x). \tag{14}$$

Conditions (13) and (14) together imply that

$$\mid g(x_0) - g(x) \mid \preceq 2d(x_0, x).$$

Let $\epsilon \succ \theta$ be given. We choose $c = \frac{\epsilon}{2} \succ \theta$. Then,

$$| g(x_0) - g(x) | \prec \epsilon \text{ whenever } d(x_0, x) \prec c.$$

This proves that g is uniformly continuous.

By an argument similar to that used above, we can compute that

$$g(x_0) + g(x) = o - \lim[2d_v(x_n, x_0) + 2d_v(x_n, x)] \succeq 2d_v(x_0, x).$$

This gives that

$$\frac{1}{2}[g(x_0) + g(x)] \succeq d_v(x_0, x).$$

Now,

$$\begin{aligned} g(x_0) - g(x) &= \frac{1}{2}\left[g(x_0) + g(x)\right] + \frac{1}{2}\left[g(x_0) - 3g(x)\right] \\ &\succeq d_v(x_0, x) + \frac{1}{2}\left[g(x_0) - 3g(x)\right]. \end{aligned} \tag{15}$$

If $g(x_0) - g(x) \succ \theta$, then $g(x) \prec g(x_0)$. Again, $g(x) \succ \theta$. So, it must be the case that $\theta \prec g(x_0)$.

It now follows from condition (12) that

$$| 2d_v(x_n, x) - g(x) | \preceq b_n \text{ for all } n.$$

This ensures that

$$g(x) - 2d_v(x_n, x) \preceq | 2d_v(x_n, x) - g(x) | \preceq b_n \text{ for all } n.$$

This being true for any $x \in X$, we have

$$g(x_{n+p}) \preceq b_n + 2d_v(x_n, x_{n+p}) \text{ for all } n \text{ and } p. \tag{16}$$

The sequence (x_n) being E-Cauchy, there is a sequence (ζ_n) in E satisfying $\zeta_n \downarrow \theta$ such that

$$d_v(x_n, x_{n+p}) \preceq \zeta_n \text{ for all } n \text{ and } p.$$

Therefore, we obtain from condition (16) that

$$3g(x_{n+p}) \preceq 3(b_n + 2\zeta_n) = c_n \text{ for all } n \text{ and } p, \tag{17}$$

where $c_n = 3(b_n + 2\zeta_n) \downarrow \theta$.

We now claim that there exists an $n_0 \in \mathbb{N}$ such that $c_{n_0} \prec g(x_0)$. If possible, suppose that $c_n \succeq g(x_0)$ for all n. Then, $\inf c_n \succeq g(x_0) \succ \theta$, which contradicts the fact that $c_n \downarrow \theta$. Thus, our claim is justified. Since the sequence (c_n) is decreasing, it follows that $c_n \prec g(x_0)$ for all $n \geq n_0$. In view of condition (17), we get

$$3g(x_{n+p}) \prec g(x_0) \text{ for all } n \geq n_0 \text{ and } p.$$

Thus, $g(x_0) - 3g(x) \succ \theta$ if $x = x_{n+p} (\neq x_0)$ and $n \geq n_0$. It now follows from condition (15) that $g(x_0) - g(x) \succ d(x_0, x)$ if $x = x_{n+p}$ and $n \geq n_0$. So, x_0 is not a d_v-point for g. □

The following corollary is the main result of J. D. Weston [20].

Corollary 3.1. *If the metric space (X, d) is complete then any lower semicontinuous function $X \to \mathbb{R}$ which is bounded below has a d-point. If (X, d) is not complete there is a uniformly continuous function $X \to \mathbb{R}$ which is bounded below but has no d-point.*

Proof. The result follows from Theorem 3.1 by taking $E = \mathbb{R}$ with usual metric. □

Remark 3.1. *Theorem* 3.1 *is an extension of the main result of Weston* [20] *in metric spaces to vector metric spaces.*

The following two examples ensure the validity of our main Theorem 3.1.

Example 3.2. *Let $E = F_c[0,1]$ be the space of all real valued constant functions defined on $[0,1]$. For $f, g \in F_c[0,1]$, we define $f \preceq g$ if and only if $f(x) \leq g(x)$ for all $x \in [0,1]$. Then $F_c[0,1]$ is a Riesz space which is Dedekind complete and totally ordered. Let $X = [0,1]$ and choose a fixed $T \in F_c[0,1]$, where $T(x) = k > 0$ for all $x \in [0,1]$. We define $d_v : X \times X \to F_c[0,1]$ by*

$$d_v(x,y) = \mid x - y \mid T \text{ for all } x,\, y \in X.$$

Then (X, d_v, E) is an E-complete vector metric space. Let $h : X \to E$ be defined by $h(x) = -\frac{x}{5}T$ for all $x \in X$. Then, $h(x) = -\frac{x}{5}T \succeq -\frac{1}{5}T$ for all $x \in X$ which ensures that h is bounded from below. Moreover, it is easy to verify that h is a lower semicontinuous function. We now compute that

$$h(0) - h(x) = \frac{x}{5}T = \frac{1}{5}d_v(0,x) \prec d_v(0,x)$$

for every $x \in X$ with $x \neq 0$. Also, $h(0) - h(x) \succ O$, zero function in E for $x \neq 0$. Thus, all the hypotheses of the first part of Theorem 3.1 *holds good and we observe that* 0 *is a d_v-point for h.*

Example 3.3. *Let $E = \mathbb{R}$ with usual metric. Then $\mathbb{R}$ is Dedekind complete and totally ordered with usual " $\leq$ ". Let $X = (0,1)$ and $d_v : X \times X \to E$ be defined by*

$$d_v(x,y) = \mid x - y \mid \text{ for all } x,\, y \in X.$$

Then (X, d_v, E) is not an E-complete vector metric space. Let $g : X \to E$ be defined by $g(x) = 2x$ for all $x \in X$. Clearly, g is uniformly continuous and bounded from below. Thus, we have all the hypotheses of the last part of Theorem 3.1*.*

We now show that g has no d_v-point. If possible, suppose that $x_0 \in X$ is a d_v-point for g. Then for any $x \in X,\ x \neq x_0$ with $g(x_0) - g(x) > 0$, we have

$$g(x_0) - g(x) < d_v(x_0, x).$$

This implies that,

$$2(x_0 - x) < \mid x_0 - x \mid . \tag{18}$$

But, $g(x_0) - g(x) > 0 \Rightarrow x_0 > x$. It now follows from (18) that $2(x_0 - x) < \mid x_0 - x \mid = x_0 - x \Rightarrow x_0 < x$, a contradiction. This proves that g has no d_v-point.

4. An Application in Fixed Point Theory

In this section we give an application of our main Theorem 3.1 in fixed point theory. We assume that (X, d_v, E) is a vector metric space, E is totally ordered and $h : X \to E$ is a function.

Remark 4.1. *When d_v and h are given, a relation " $\ll$ " can be defined on X as follows:*

$$x \ll y \text{ if and only if } h(y) - h(x) \succeq d_v(x,y).$$

This relation orders X. In fact, " $\ll$ " is reflexive, antisymmetric and transitive.

Definition 4.1. *A point x_0 in (X, d_v, E) is said to be a minimal point w.r.t. $\ll$ if and only if $x \ll x_0$ implies $x = x_0$.*

Theorem 4.1. *A point of X is a d_v-point for h if and only if it is a minimal point w.r.t. $\ll$.*

Proof. Let $x_0 \in X$ be a d_v-point for h. Then, for every other point $x(\neq x_0) \in X$ with $h(x_0) - h(x) \succ \theta$,

$$h(x_0) - h(x) \prec d_v(x_0, x). \tag{19}$$

Now $x \ll x_0$ implies that $h(x_0) - h(x) \succeq d_v(x, x_0)$. This gives that $x = x_0$. Because if $x \neq x_0$, then $h(x_0) - h(x) \succeq d_v(x, x_0) \succ \theta$. So it follows from condition (19) that $h(x_0) - h(x) \prec d_v(x, x_0)$, which is a contradiction. Therefore, x_0 is a minimal point w.r.t. $\ll$.

Conversely, let x_0 be a minimal point w.r.t. $\ll$. Then $x \ll x_0$ implies that $x = x_0$. That is, $x \ll x_0$ does not hold for any $x \in X$ with $x \neq x_0$. As E is totally ordered, we have $h(x_0) - h(x) \prec d_v(x, x_0)$ for all $x \in X$ with $x \neq x_0$. This gives that x_0 is a d_v-point for h. □

Theorem 4.2. *If a function $f : X \to X$ is such that it may be possible to choose d_v and h so that the relation $\ll$ has the property that $fx \neq x$ implies $fx \ll x$, then any d_v-point for h is a fixed point for f.*

Proof. Let $x_0 \in X$ be a d_v-point for h. Then, for every other point $x(\neq x_0) \in X$ with $h(x_0) - h(x) \succ \theta$,

$$h(x_0) - h(x) \prec d_v(x, x_0). \tag{20}$$

If $fx_0 \neq x_0$, then by hypothesis $fx_0 \ll x_0$ which implies that

$$h(x_0) - h(fx_0) \succeq d_v(fx_0, x_0) \succ \theta,$$

which contradicts the condition (20). So, it must be the case that $fx_0 = x_0$. This shows that x_0 is a fixed point of f. □

We now apply Theorems 3.1 and 4.2 to prove analogue of Banach Contraction Principle in vector metric spaces.

Theorem 4.3. *Suppose that E is a Riesz space which is Dedekind complete and totally ordered. Let (X, d_v, E) be an E-complete vector metric space and let $f : X \to X$ be a mapping satisfying the following condition:*

$$d_v(fx, fy) \preceq \alpha\, d_v(x, y) \tag{21}$$

for all x, $y \in X$, where $0 \leq \alpha < 1$ is a constant. Then f has a unique fixed point in X.

Proof. Let $h(x) = \beta\, d_v(fx, x)$, where $\beta = \frac{1}{1-\alpha} > 0$ and $x \in X$. It is obvious that h is bounded from below. We first show that $h : X \to E$ is a lower semicontinuous function. Let $y_n \xrightarrow{d_v, E} y$ in (X, d_v, E). We have to show that

$$h(y) \preceq \liminf_{n\to\infty} h(y_n).$$

By using condition (21), we have

$$\begin{aligned} h(y) = \beta\, d_v(fy, y) &\preceq \beta\, [d_v(fy, y_n) + d_v(y_n, y)] \\ &\preceq \beta\, [d_v(fy, fy_n) + d_v(fy_n, y_n) + d_v(y_n, y)] \\ &\preceq \beta\, [\alpha\, d_v(y, y_n) + d_v(fy_n, y_n) + d_v(y_n, y)] \\ &= \beta(\alpha + 1)\, d_v(y, y_n) + h(y_n). \end{aligned}$$

This gives that,

$$h(y) \preceq \liminf_{n\to\infty} h(y_n).$$

Thus, h is a lower semicontinuous function on an E-complete vector metric space (X, d_v, E). Therefore, Theorem 3.1 ensures the existence of a d_v-point u(say) for h.

We now show that $fx \neq x$ implies $fx \ll x$.

Let $fx \neq x$. By using condition (21), we obtain

$$\begin{aligned} h(x) - h(fx) &= \beta\,[d_v(fx,x) - d_v(f^2x, fx)] \\ &\succeq \beta\,[d_v(fx,x) - \alpha\, d_v(fx,x)] \\ &= \beta\,(1-\alpha)\, d_v(fx,x)] \\ &= d_v(fx,x). \end{aligned}$$

Thus f satisfies the condition that $fx \neq x$ implies $fx \ll x$. By applying Theorem 4.2, it follows that the d_v-point u for h is a fixed point for f in X.

For uniqueness, let $v \in X$ be another fixed point of f. Then, by condition (21), we get

$$d_v(u,v) = d_v(fu, fv) \preceq \alpha\, d_v(u,v).$$

Since $0 \leq \alpha < 1$, it follows from Lemma 2.1 that $d_v(u,v) = \theta$ and hence $u = v$. □

The following example supports our Theorem 4.3.

Example 4.1. *We consider $E = F_c[0,1]$ as in Example 3.2. Then E is a Riesz space which is Dedekind complete and totally ordered. Let $X = [0,\infty)$ and choose a fixed $T \in E$, where $T(x) = k > 0$ for all $x \in [0,1]$. We define $d_v : X \times X \to E$ by*

$$d_v(x,y) = \mid x - y \mid T \text{ for all } x,\, y \in X.$$

Then (X, d_v, E) is an E-complete vector metric space. Let $f : X \to X$ be a mapping defined by $fx = \frac{x+1}{2}$ for all $x \in X$. We now compute that

$$d_v(fx, fy) = \mid fx - fy \mid T = \frac{1}{2} \mid x - y \mid T = \frac{1}{2} d_v(x,y) \text{ for all } x,\, y \in X.$$

Thus, all the conditions of Theorem 4.3 holds true and we observe that 1 is the unique fixed point of f in X.

Remark 4.2. *It is worthy to note that the well known Banach Contraction Principle [4] in metric spaces can be obtained from Theorem 4.3 by taking $E = \mathbb{R}$.*

5. Conclusions

In this study, we characterized E-completeness in vector metric spaces in terms of d_v-point. As a consequence of this study, we have been able to establish an important fixed point theorem in vector metric spaces. Our main result extends the result of Weston [20].

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Sushanta Kumar Mohanta received his Ph.D. degree in Mathematics from The University of Burdwan, West Bengal, India. He is a Professor of the Department of Mathematics, West Bengal State University, Kolkata. His research interest includes Fixed Point Theory and its applications, Operator Theory, Functional Analysis, Bitopological spaces.

Shubha Das received M.Sc. in Mathematics from West Bengal State University in 2018. He is a research fellow of the Department of Mathematics, West Bengal State University, Kolkata [JRF Award letter No. NSFDC/E-88929, Date: 08-Oct-2024]. His research interest includes Fixed Point Theory and its applications, Operator Theory.