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HIGHER-ORDER ACCURATE COMPACT SCHEMES AND ANALYSIS FOR THE TIME-FRACTIONAL BLACK-SCHOLES MODEL

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ABSTRACT. This work is focused on the construction of numerical schemes with higher-order accuracy in space and time to solve the time-fractional Black-Scholes model that governs the price of European options. We develop three numerical schemes utilizing the fourth-order Padé approximation, a fourth-order Taylor's compact difference scheme and a fourth-order compact exponential scheme for spatial discretization. We employ $L1$-2-3 approximation of order $4-\alpha,\ 0<\alpha<1$, to discretize the time-fractional derivative. In addition, the solvability, convergence, and stability of these numerical schemes are established. Numerical experiments are conducted to demonstrate the accuracy of the proposed schemes and validate the theoretical findings. The new proposed schemes offer higher and better accuracy.

Keywords: Time-fractional Black-Scholes model, Padé approximation, Taylor's compact difference, compact exponential, solvability, stability, convergence.

AMS Subject Classification: 35R11, 65M12, 65M06.

1. Introduction

Options are highly liquid financial instruments that are often traded in the market. Options pricing has gained significant attention and can be traced back to the Black-Scholes (B-S) model, which was introduced in 1973 by Black and Scholes [2] and Merton [14]. The application of fractional derivatives and integrals is witnessing considerable expansion due to their ability to effectively integrate historical data, attributable to their nonlocal properties [11]. Furthermore, there is a growing emergence of distributed order fractional equations [16], in which the fractional order is a continuous spectrum. Fractional calculus has provided a powerful tool for modeling anomalous behaviors in various fields of science and engineering, such as material science, biology, physics, control science, finance and fluid mechanics [8]. Some of the numerical methods used to solve fractional differential

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equations include finite difference methods, finite element methods, finite volume methods, spectral methods and collocation methods [5, 17, 21].

Identifying the fractal nature of a stochastic process has led to the integration of fractional calculus into stochastic models and financial theory. Wyss [25] used a time-fractional B-S model to determine the price of a European call option. Liang et al. formulated a single parameter and a biparameter fractional Black-Scholes-Merton differential equation in [12], assuming that a fractional Itô process may describe the movement of the stock price. For the time-fractional Black-Scholes (TFBS) model, Zhang et al. [26] constructed a numerical scheme with order of accuracy $2-\alpha$ in time and second-order spatial accuracy, and analyzed stability and convergence. Later, De Staelen and Hendy [6] constructed a higher-order difference method with fourth-order spatial accuracy and proved the stability and convergence. Tian et al. [24] presented three different compact finite difference schemes for the TFBS model governing European option pricing, in which the spatial convergence accuracy of these three algorithms is fourth-order, and the temporal accuracy orders are $2-\alpha$, 2 and $3-\alpha$ respectively. In [7], Huang et al. used a compact exponential scheme for space and the $L1$ formula for discretizing the time-fractional derivative. Roul [18] presented a collocation method based on the quintic B-spline basis function with convergence order $2-\alpha$ in time and fourth-order in space. An et al. [1] presented a space-time spectral method and analyzed the stability and convergence of the numerical method. Song and Lyu [19] investigated a fast and high-order numerical method for the TFBS equation, with a weak initial singularity of the solution. Cai and Wang [3] proposed a novel methodology utilizing a tailored finite point method for spatial discretization and $L1$-discretization for the time-fractional derivative. Taghipour and Aminikhah [20] proposed an efficient spectral collocation method. Kaur and Natesan [9] proposed a numerical method based on the cubic spline method. Zhang and Zheng [27] proposed the finite element method to solve the variable order TFBS model. Kazmi [10] developed an efficient numerical method with second-order temporal and spatial accuracy.

The time-fractional Black-Scholes model [4], is given by

$$\begin{cases} \dfrac{\partial^\alpha U(S,\tau)}{\partial \tau^\alpha} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 U(S,\tau)}{\partial S^2} + rS\dfrac{\partial U(S,\tau)}{\partial S} - rU(S,\tau) = 0, \; (S,\tau) \in (0,\infty)\times[0,T), \\ U(S,T) = z(S), \\ U(0,\tau) = p(\tau), \quad \lim\limits_{S\to\infty} U(S,\tau) = q(\tau), \end{cases} \tag{1}$$

where $\alpha \in (0,1)$. When $\alpha = 1$ the model (1) reduces to the classical B-S model. Here, $U(S,\tau)$ denotes the price of an option with S being the price of the asset and τ the current time, $r > 0$ the risk-free interest rate, $\sigma > 0$ the volatility of the underlying asset and $T > 0$ the expiration time.

In this work, we develop three new compact numerical schemes to solve the time fractional Black-Scholes (TFBS) model. The spatial derivatives are approximated using the fourth-order Padé approximation, a fourth-order Taylor compact difference scheme and a fourth-order compact exponential scheme. We improve the accuracy by adopting the $L1$-2-3 discretization of order $4-\alpha$, to approximate the time-fractional derivative. The solvability of these numerical schemes is established. The stability and convergence analysis of the proposed schemes is proved using Fourier analysis. Numerical experiments are conducted, and the results are compared with some existing methods in the literature.

The paper is organized as follows. Section 1 presents a detailed literature survey on numerical schemes to solve the TFBS model. In Section 2, an equivalent model of the

TFBS is formulated. Three numerical schemes are constructed in Section 3 and their solvability is analyzed. The stability and convergence analysis is presented in Section 4. The numerical results are demonstrated in Section 5. Finally, Section 6 concludes the study by summarizing the key findings.

2. Time fractional Black-Scholes model

Using the transformations [24],

$$S = e^x,\ \tau = T - t,\ V(x,t) = U\left(e^x, T-t\right),$$

the model (1) can be written as

$$\begin{cases} {}_0^C D_t^\alpha V(x,t) - \dfrac{1}{2}\sigma^2 V_{xx}(x,t) - \left(r - \dfrac{1}{2}\sigma^2\right) V_x(x,t) + rV(x,t) = 0, \\ \hfill (x,t) \in (-\infty,\infty) \times (0,T], \\ V(x,0) = z(x), \\ \lim\limits_{x\to-\infty} V(x,t) = p(t),\ \lim\limits_{x\to\infty} V(x,t) = q(t). \end{cases} \tag{2}$$

In model (1), $\dfrac{\partial^\alpha U(S,\tau)}{\partial \tau^\alpha}$ is transformed to the following Caputo form [26]:

$${}_0^C D_t^\alpha V(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial V(x,\zeta)}{\partial \zeta} (t-\zeta)^{-\alpha} \mathrm{d}\zeta.$$

To solve the above model numerically, we need to truncate the unbounded domain $(-\infty,\infty) \times (0,T]$ into a finite interval $(b_d, b_u) \times (0,T]$ and add a source term $f(x,t)$ to the right-hand side of the equation without loss of generality. The model (2) can be formulated in the following form [18]:

$$\begin{cases} {}_0^C D_t^\alpha V(x,t) - \dfrac{1}{2}\sigma^2 V_{xx}(x,t) - \left(r - \dfrac{1}{2}\sigma^2\right) V_x(x,t) + rV(x,t) = f(x,t), \\ \hfill b_d < x < b_u,\ 0 < t \le T, \\ V(x,0) = z(x),\ b_d < x < b_u, \\ V(b_d,t) = p(t),\ V(b_u,t) = q(t),\ 0 \le t \le T. \end{cases} \tag{3}$$

3. Construction of numerical schemes

In this section, three numerical schemes with fourth-order accuracy in space and time accuracy of order $4-\alpha$ are developed to solve the TFBS model. In the beginning, we give notations and some preliminaries. Let $t_n = n\Delta\tau,\ n = 0,1,2,\ldots,N;\ x_i = b_d + ih,\ i = 0,1,2,\ldots,M$ represent the uniform time and spatial meshes, respectively. Here, $\Delta\tau = \dfrac{T}{N}$ and $h = \dfrac{b_u - b_d}{M}$ denote the time and spatial grid sizes, respectively. The following notations are introduced:

$$\begin{aligned} \delta_x V\left(x_i, t_n\right) &= \frac{V\left(x_{i+1}, t_n\right) - V\left(x_{i-1}, t_n\right)}{2h}, \\ \delta_x^2 V\left(x_i, t_n\right) &= \frac{V\left(x_{i+1}, t_n\right) - 2V\left(x_i, t_n\right) + V\left(x_{i-1}, t_n\right)}{h^2}, \\ \mathcal{A} V\left(x_i, t_n\right) &= \frac{1}{12}\left[V\left(x_{i-1}, t_n\right) + 10V\left(x_i, t_n\right) + V\left(x_{i+1}, t_n\right)\right]. \end{aligned}$$

Let $f \in C^4[0, t_k]$. For any α $(0 < \alpha < 1)$, $L1$-2-3 approximation [15] of ${}_0^C D_t^\alpha f(t)\big|_{t=t_k}$ is defined as

$$\widetilde{\mathbb{D}}_t^\alpha f(t)\Big|_{t=t_k} = \frac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)} \left(\gamma_0^\alpha f_k - \sum_{j=1}^{k-1} \left(\gamma_{k-j-1}^\alpha - \gamma_{k-j}^\alpha\right) f_j - \gamma_{k-1}^\alpha f_0 \right), \tag{4}$$

where, for $k = 1, \gamma_0^\alpha = 1$; for $k = 2, \gamma_0^\alpha = a_0^\alpha + b_0^\alpha, \gamma_1^\alpha = a_1^\alpha - b_0^\alpha$;

for $k = 3$,

$$\gamma_l^\alpha = \begin{cases} a_l^\alpha + b_l^\alpha + \beta_l^\alpha, & l = 0, \\ a_l^\alpha + b_l^\alpha - b_{l-1}^\alpha - 2\beta_{l-1}^\alpha, & l = 1, \\ a_l^\alpha - b_{l-1}^\alpha + \beta_{l-2}^\alpha, & l = 2, \end{cases}$$

and for $k \geq 4$,

$$\gamma_l^\alpha = \begin{cases} a_l^\alpha + b_l^\alpha + \beta_l^\alpha, & l = 0, \\ a_l^\alpha + b_l^\alpha - b_{l-1}^\alpha + \beta_l^\alpha - 2\beta_{l-1}^\alpha, & l = 1, \\ a_l^\alpha + b_l^\alpha - b_{l-1}^\alpha + \beta_l^\alpha - 2\beta_{l-1}^\alpha + \beta_{l-2}^\alpha, & l \leq k-3, \\ a_l^\alpha + b_l^\alpha - b_{l-1}^\alpha - 2\beta_{l-1}^\alpha + \beta_{l-2}^\alpha, & l = k-2, \\ a_l^\alpha - b_{l-1}^\alpha + \beta_{l-2}^\alpha, & l = k-1, \end{cases}$$

where

$$a_l^\alpha = (l+1)^{1-\alpha} - l^{1-\alpha}, \ l \geq 0,$$
$$b_l^\alpha = \left[(l+1)^{2-\alpha} - l^{2-\alpha}\right]/(2-\alpha) - \left[(l+1)^{1-\alpha} + l^{1-\alpha}\right]/2, \ l \geq 0,$$
$$\beta_l^\alpha = -\left(\frac{1}{6}\left((l+1)^{1-\alpha} + 2l^{1-\alpha}\right) + \frac{1}{2-\alpha} l^{2-\alpha} - \frac{1}{(2-\alpha)(3-\alpha)}\left((l+1)^{3-\alpha} - l^{3-\alpha}\right) \right), \ l \geq 0.$$

Lemma 3.1. [15] *For any* α *and* γ_j^α $(0 \leq j \leq k-1, \ k \geq 4)$ *defined in (4), we have*

(i) $\gamma_0^\alpha = \dfrac{1}{3} + \dfrac{1}{2-\alpha} + \dfrac{1}{(2-\alpha)(3-\alpha)} \in \left(1, \dfrac{11}{6}\right)$,

(ii) $\gamma_0^\alpha > |\gamma_1^\alpha|$,

(iii) $\gamma_j^\alpha > 0, j \neq 1$,

(iv) $\gamma_2^\alpha \geq \gamma_3^\alpha \geq \cdots \geq \gamma_{k-3}^\alpha \geq \gamma_{k-2}^\alpha \geq \gamma_{k-1}^\alpha$,

(v) $\gamma_0^\alpha > \gamma_2^\alpha$,

(vi) $\displaystyle\sum_{j=0}^{k-1} \gamma_j^\alpha = k^{1-\alpha}$.

Theorem 3.1. [15] *Let* $f \in C^4[0,t_k]$ *and* $R(f(t_k)) = {}_0^C D_t^\alpha f(t)\big|_{t=t_k} - \widetilde{\mathbb{D}}_t^\alpha f(t_k)$, *for* α $(0<\alpha<1)$. *Then, we have*

$$|R(f(t_1))| \le \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_0\le t\le t_1} \left|f''(t)\right| \Delta\tau^{2-\alpha},$$

$$\begin{aligned}|R(f(t_2))| \le \frac{1}{\Gamma(1-\alpha)} \Bigg\{ & \frac{\alpha}{12} \max_{t_0\le t\le t_1} \left|f''(t)\right| (t_2-t_1)^{-\alpha-1} \Delta\tau^3 \\ & + \left[\frac{1}{12} + \frac{\alpha}{3(1-\alpha)(2-\alpha)}\left(\frac{1}{2}+\frac{1}{3-\alpha}\right)\right] \max_{t_0\le t\le t_2} \left|f^{(3)}(t)\right| \Delta\tau^{3-\alpha} \Bigg\},\end{aligned}$$

$$\begin{aligned}|R(f(t_k))| \le \frac{\alpha}{\Gamma(1-\alpha)} \Bigg\{ & 12(t_k-t_1)^{-\alpha-1} \max_{t_0\le t\le t_1} \left|f''(t)\right| \Delta\tau^3 \\ & + \frac{1}{8}(t_k-t_2)^{-\alpha-1} \max_{t_0\le t\le t_2} \left|f^{(3)}(t)\right| \Delta\tau^4 \\ & + \left(\frac{1}{2} + \frac{1}{12}\frac{\alpha^2-10\alpha+27}{\prod_{i=1}^4(\alpha-i)}\right) \max_{t_0\le t\le t_k} \left|f^{(4)}(t)\right| \Delta\tau^{4-\alpha} \Bigg\}, \text{ for } k\ge 3.\end{aligned}$$

3.1. $L1$**-2-3-Padé Scheme.** Multiplying (3) with $\dfrac{2}{\sigma^2}$, we get

$$\frac{2}{\sigma^2}{}_0^C D_t^\alpha V(x,t) - V_{xx}(x,t) + \left(1-\frac{2r}{\sigma^2}\right)V_x(x,t) + \frac{2r}{\sigma^2}V(x,t) = \frac{2}{\sigma^2}f(x,t). \tag{5}$$

Let $1-\dfrac{2r}{\sigma^2}=\beta$. Using the exponential transformation $V(x,t)=e^{\frac{1}{2}\beta x}u(x,t)$, given in [13], the equation (5) is transformed to

$${}_0^C D_t^\alpha u(x,t) - \frac{\sigma^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + \left[\frac{1}{2\sigma^2}\left(r-\frac{\sigma^2}{2}\right)^2 + r\right]u(x,t) = f(x,t)\; e^{-\frac{1}{2}\beta x}.$$

For convenience, let $\dfrac{\sigma^2}{2}=s$ and $\dfrac{1}{2\sigma^2}\left(r-\dfrac{\sigma^2}{2}\right)^2+r=w$. Now, it is easy to see that $s>0,\ w>0$. Therefore, the model (3) can be represented as follows:

$${}_0^C D_t^\alpha u(x,t) - s\frac{\partial^2 u(x,t)}{\partial x^2} + wu(x,t) = g(x,t),\ b_d<x<b_u,\ 0<t\le T, \tag{6}$$

with the initial and boundary conditions:

$$\begin{aligned}&u(x,0)=z(x)\,e^{-\frac{1}{2}\beta x},\ b_d<x<b_u,\\ &u(b_d,t)=p(t)\,e^{-\frac{1}{2}\beta b_d},\ u(b_u,t)=q(t)\,e^{-\frac{1}{2}\beta b_u},\ 0\le t\le T,\end{aligned}$$

where $g(x,t)=f(x,t)\,e^{-\frac{1}{2}\beta x}$.

Considering (6) at the grid point (x_i,t_n), we have

$${}_0^C D_t^\alpha u(x_i,t_n) - s\frac{\partial^2 u(x_i,t_n)}{\partial x^2} + wu(x_i,t_n) = g(x_i,t_n),\ 1\le i\le M-1;\ 1\le n\le N. \tag{7}$$

Applying $L1$-2-3 approximation (4), ${}_0^C D_t^\alpha u(x_i,t_n)$ is approximated as

$${}_0^C D_t^\alpha u(x_i,t_n) = \frac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(\gamma_0^\alpha u(x_i,t_n) - \sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha-\gamma_{n-k}^\alpha\right)u(x_i,t_k) - \gamma_{n-1}^\alpha u(x_i,t_0)\right). \tag{8}$$

Using the Padé scheme, the second-order spatial derivative is approximated as

$$\frac{\partial^2 u\left(x_i,t_n\right)}{\partial x^2}=\frac{\delta_x^2}{\mathcal{I}+\frac{h^2}{12}\delta_x^2}u\left(x_i,t_n\right)+O\left(h^4\right). \tag{9}$$

Let $\mathcal{A}=\mathcal{I}+\dfrac{h^2}{12}\delta_x^2$. Substituting (8) and (9) into (7), we obtain

$$\begin{aligned}&\mathcal{A}\left\{\frac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[\gamma_0^\alpha u\left(x_i,t_n\right)-\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha-\gamma_{n-k}^\alpha\right)u\left(x_i,t_k\right)-\gamma_{n-1}^\alpha u\left(x_i,t_0\right)\right]\right\}\\&\quad=s\delta_x^2 u\left(x_i,t_n\right)-w\mathcal{A}u\left(x_i,t_n\right)+\mathcal{A}g\left(x_i,t_n\right)+\hat{r}_i^n,\ 1\le i\le M-1;\ 1\le n\le N.\end{aligned} \tag{10}$$

There exists a positive constant C_1 such that

$$\left|\hat{r}_i^n\right|\le C_1\left(\Delta\tau^{4-\alpha}+h^4\right),\ 1\le i\le M-1;\ 3\le n\le N.$$

Denoting $\hat{u}_i^n$ as the approximate solution of $u\left(x_i,t_n\right)$, omitting the term $\hat{r}_i^n$ and rearranging the terms in (10), the L1-2-3-Padé scheme for (6) with initial and boundary conditions is obtained as follows:

$$\begin{aligned}&\left(\frac{\mu}{12}\gamma_0^\alpha-\frac{s}{h^2}+\frac{w}{12}\right)\hat{u}_{i-1}^n+\left(\frac{10\mu}{12}\gamma_0^\alpha+\frac{2s}{h^2}+\frac{10w}{12}\right)\hat{u}_i^n+\left(\frac{\mu}{12}\gamma_0^\alpha-\frac{s}{h^2}+\frac{w}{12}\right)\hat{u}_{i+1}^n\\&\quad=\frac{\mu}{12}\sum_{k=1}^{n-1}\left[\gamma_{n-k-1}^\alpha-\gamma_{n-k}^\alpha\right]\left(\hat{u}_{i-1}^k+10\hat{u}_i^k+\hat{u}_{i+1}^k\right)+\frac{\mu}{12}\gamma_{n-1}^\alpha\left(\hat{u}_{i-1}^0+10\hat{u}_i^0+\hat{u}_{i+1}^0\right)\\&\qquad+\frac{1}{12}\left(g_{i+1}^n+10g_i^n+g_{i-1}^n\right),\ i=1,2,\dots,M-1;\ n=1,2,\dots,N,\\[2ex]&\hat{u}_i^0=z(x_i)\,e^{-\frac{1}{2}\beta x_i},\ i=1,2,\dots,M-1,\\&\hat{u}_0^n=p(t_n)\,e^{-\frac{1}{2}\beta b_d},\ \ \hat{u}_M^n=q(t_n)\,e^{-\frac{1}{2}\beta b_u},\ \ n=0,1,\dots,N,\end{aligned} \tag{11}$$

where $\mu=\dfrac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}$.

Theorem 3.2. *The L1-2-3-Padé scheme (11) has a unique solution.*

Proof. The difference scheme (11) can be written in a more concise form

$$\mathrm{A}\hat{u}^n=\mathbf{b},$$

where the right-hand side vector $\mathbf{b}$ depends on the solution $\hat{u}^{n-1},\hat{u}^{n-2},\dots,\hat{u}^0$ and the tri-diagonal matrix $\mathrm{A}=(a_{ij})$, where

$$|a_{ii}|=\frac{10\mu}{12}\gamma_0^\alpha+\frac{2s}{h^2}+\frac{10w}{12}\ \forall\, i=1,2,\dots,M-1 \text{ and}$$

$$\sum_{j\ne i}|a_{ij}|=\begin{cases}\dfrac{2\mu}{12}\gamma_0^\alpha-\dfrac{2s}{h^2}+\dfrac{2w}{12}\ \text{or}\ -\dfrac{2\mu}{12}\gamma_0^\alpha+\dfrac{2s}{h^2}-\dfrac{2w}{12}\ \text{if } i=2,3,\dots,M-2,\\ \dfrac{\mu}{12}\gamma_0^\alpha-\dfrac{s}{h^2}+\dfrac{w}{12}\ \text{or}\ -\dfrac{\mu}{12}\gamma_0^\alpha+\dfrac{s}{h^2}-\dfrac{w}{12}\ \text{if } i=1,M-1.\end{cases}$$

Since μ, s, w and γ_0^α are nonnegative, and $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$, the tri-diagonal coefficient matrix A $= (a_{ij})$ is strictly diagonally dominant. Thus, A is non-singular. Hence, there exists a unique solution. □

3.2. L1-2-3-**Taylor's compact scheme (**L1-2-3-**TCS).** Consider

$$\begin{cases} a\dfrac{\partial^2 V(x,t)}{\partial x^2} + b\dfrac{\partial V(x,t)}{\partial x} = {}_0^C D_t^\alpha V(x,t) + cV(x,t) - f(x,t), \quad (x,t) \in (b_d, b_u) \times (0,T], \\ V(x,0) = z(x),\ b_d < x < b_u, \\ V(b_d,t) = p(t),\ V(b_u,t) = q(t),\ 0 \le t \le T, \end{cases} \tag{12}$$

where $a = \dfrac{1}{2}\sigma^2 > 0,\ b = r - a,\ c = r > 0.$

Using Taylor's expansion, a fourth order semi-discrete compact scheme for (12), see [6], can be obtained as

$$\begin{aligned} \left(a + \frac{h^2 b^2}{12a}\right)\delta_x^2 V(x_i,t_n) + b\delta_x V(x_i,t_n) = \frac{h^2}{12}\left(\delta_x^2 g(x_i,t_n) + \frac{b}{a}\delta_x g(x_i,t_n)\right) \\ + g(x_i,t_n) + O\left(h^4\right), \end{aligned} \tag{13}$$

with $g(x_i,t_n) = {}_0^C D_t^\alpha V(x_i,t_n) + cV(x_i,t_n) - f(x_i,t_n)$. The Caputo time-fractional derivative in $g(x_i,t_n)$ is approximated by using the L1-2-3 discretization (4). Using (4) in (13) and rearranging the terms, we obtain

$$\begin{aligned} &\left(a + \frac{h^2 b^2}{12a}\right)\delta_x^2 V(x_i,t_n) + b\delta_x V(x_i,t_n) \\ &= \frac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(\gamma_0^\alpha V(x_i,t_n) - \sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)V(x_i,t_k) - \gamma_{n-1}^\alpha V(x_i,t_0)\right) + cV(x_i,t_n) \\ &- f(x_i,t_n) + \frac{h^2}{12}\left(\delta_x^2 + \frac{b}{a}\delta_x\right)\left[\frac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(\gamma_0^\alpha V(x_i,t_n) - \sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)V(x_i,t_k)\right.\right. \\ &\left.\left.- \gamma_{n-1}^\alpha V(x_i,t_0)\right) + cV(x_i,t_n) - f(x_i,t_n)\right] + R_i^n,\ 1 \le i \le M-1;\ 1 \le n \le N. \end{aligned} \tag{14}$$

There exists a positive constant C_2 such that

$$|R_i^n| \le C_2\left(\Delta\tau^{4-\alpha} + h^4\right),\ 1 \le i \le M-1;\ 3 \le n \le N. \tag{15}$$

In (14), denoting $v_i^n{=}V(x_i,t_n)$, omitting the term R_i^n and after rearranging the terms, the $L1$-2-3-TCS for the problem (12) is obtained as:

$$\begin{aligned}
&\left[\mu_1-\frac{\mu\gamma_0^\alpha+c}{12}-\frac{b}{2h}+\frac{bh}{24a}(\mu\gamma_0^\alpha+c)\right]v_{i-1}^n+\left[-2\mu_1-\frac{10}{12}(\mu\gamma_0^\alpha+c)\right]v_i^n\\
&+\left[\mu_1-\frac{\mu\gamma_0^\alpha+c}{12}+\frac{b}{2h}-\frac{bh}{24a}(\mu\gamma_0^\alpha+c)\right]v_{i+1}^n\\
&=-f_i^n-\frac{1}{12}\left(f_{i-1}^n-2f_i^n+f_{i+1}^n\right)-\frac{bh}{24a}\left(f_{i+1}^n-f_{i-1}^n\right)\\
&-\mu\left(\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha-\gamma_{n-k}^\alpha\right)v_i^k+\gamma_{n-1}^\alpha v_i^0\right)\\
&-\frac{\mu}{12}\left[\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha-\gamma_{n-k}^\alpha\right)\left(v_{i-1}^k-2v_i^k+v_{i+1}^k\right)+\gamma_{n-1}^\alpha\left(v_{i-1}^0-2v_i^0+v_{i+1}^0\right)\right]\\
&-\frac{b\mu h}{24a}\left[\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha-\gamma_{n-k}^\alpha\right)\left(v_{i+1}^k-v_{i-1}^k\right)+\gamma_{n-1}^\alpha\left(v_{i+1}^0-v_{i-1}^0\right)\right],\\
&\qquad\qquad i=1,2,\dots,M-1;\; n=1,2,\dots,N,
\end{aligned}\tag{16}$$

$v_i^0=z(x_i),\;\; i=1,2,\dots,M-1,$

$v_0^n=p(t_n),\;\; v_M^n=q(t_n),\;\; n=0,1,2,\dots,N,$

where $\mu=\dfrac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}$ and $\mu_1=\dfrac{1}{h^2}\left(a+\dfrac{b^2h^2}{12a}\right)$.

Let $v^n=\left[v_1^n,v_2^n,\dots,v_{M-1}^n\right]^T$. Then, we write (16) as a system of algebraic equations $Av^n=\mathbf{b}$, where $A=\text{tridiag}(L,D,U)$, with $L=\left[\mu_1-\dfrac{\mu\gamma_0^\alpha+c}{12}-\dfrac{b}{2h}+\dfrac{bh}{24a}(\mu\gamma_0^\alpha+c)\right]$, $D=\left[-2\mu_1-\dfrac{10}{12}(\mu\gamma_0^\alpha+c)\right]$ and $U=\left[\mu_1-\dfrac{\mu\gamma_0^\alpha+c}{12}+\dfrac{b}{2h}-\dfrac{bh}{24a}(\mu\gamma_0^\alpha+c)\right]$.

Theorem 3.3. *The $L1$-2-3-TCS (16) has a unique solution.*

Proof. Since A is a Toeplitz matrix, its eigenvalues are given by

$$\lambda_i=-2\mu_1-\frac{10}{12}(\mu\gamma_o^\alpha+c)+2\sqrt{\left(\mu_1-\frac{\mu\gamma_0^\alpha+c}{12}\right)^2-\left(\frac{b}{2h}-\frac{bh}{24a}(\mu\gamma_0^\alpha+c)\right)^2}\cos\frac{i\pi}{M},\quad i=1,2,\dots,M-1.$$

For the case

$$\left(\mu_1-\frac{\mu\gamma_0^\alpha+c}{12}\right)^2-\left(\frac{b}{2h}-\frac{bh}{24a}(\mu\gamma_0^\alpha+c)\right)^2\le 0,$$

the eigenvalues of A can be written as $\lambda_i=a+b\mathrm{i}$ in which $a\neq 0$.
For the case

$$\left(\mu_1-\frac{\mu\gamma_0^\alpha+c}{12}\right)^2-\left(\frac{b}{2h}-\frac{bh}{24a}(\mu\gamma_0^\alpha+c)\right)^2>0,$$

if $\mu_1-\dfrac{\mu\gamma_0^\alpha+c}{12}>0$, we have

$$\lambda_i \leq -2\mu_1 - \frac{10}{12}(\mu\gamma_o^\alpha + c) + 2\left(\mu_1 - \frac{\mu\gamma_0^\alpha + c}{12}\right) = -(\mu\gamma_o^\alpha + c) < 0,$$

and if $\mu_1 - \dfrac{\mu\gamma_0^\alpha + c}{12} < 0$, we have

$$\lambda_i \leq -2\mu_1 - \frac{10}{12}(\mu\gamma_o^\alpha + c) - 2\left(\mu_1 - \frac{\mu\gamma_0^\alpha + c}{12}\right) = -\left(4\mu_1 + \frac{8}{12}(\mu\gamma_o^\alpha + c)\right) < 0.$$

Since μ_1, μ, c and γ_o^α are positive constants, in each case the eigenvalues of A are nonzero, and consequently, the coefficient matrix A is invertible. Therefore, the solution of $L1$-2-3-TCS (16) exists and is unique. □

3.3. $L1$-2-3-Compact exponential scheme ($L1$-2-3-CES).

First, we consider the steady state equation

$$b\frac{dV}{dx} + a\frac{d^2V}{dx^2} = F(x), \tag{17}$$

where $F(x)$ is a suitable smooth function. A fourth-order compact exponential scheme for (17), see [23], can be obtained as

$$\left(-\beta\delta_x^2 + b\delta_x\right)V_i = \left(1 + \beta_1\delta_x + \beta_2\delta_x^2\right)F_i + O\left(h^4\right), \tag{18}$$

where

$$\beta = \frac{bh}{2}\coth\left(\frac{-bh}{2a}\right),\ \beta_1 = \frac{-a-\beta}{b},\ \beta_2 = \frac{a(a+\beta)}{b^2} + \frac{h^2}{6}.$$

Substituting $F(x) = {}_0^C D_t^\alpha V(x,t) + cV(x,t) - f(x,t)$ in (18) and rearranging the terms, the compact exponential semi-discrete approximation for the TFBS problem (12) is developed as

$$\begin{aligned}
&\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right)\left({}_0^C D_t^\alpha V + cV - f\right)_{i-1}^n + \left(1 - \frac{2\beta_2}{h^2}\right)\left({}_0^C D_t^\alpha V + cV - f\right)_i^n \\
&\quad + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right)\left({}_0^C D_t^\alpha V + cV - f\right)_{i+1}^n \\
&\quad = \left(-\frac{\beta}{h^2} - \frac{b}{2h}\right)V_{i-1}^n + \frac{2\beta}{h^2}V_i^n + \left(-\frac{\beta}{h^2} + \frac{b}{2h}\right)V_{i+1}^n + O\left(h^4\right).
\end{aligned} \tag{19}$$

Using (4) in (19), and rearranging the terms, the scheme for (12) is developed as

$$\begin{aligned}
&\left[(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} + \frac{b}{2h}\right)\right]V_{i-1}^n + \left[(\mu\gamma_o^\alpha + c)\left(1 - \frac{2\beta_2}{h^2}\right) - \frac{2\beta}{h^2}\right]V_i^n \\
&+ \left[(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} - \frac{b}{2h}\right)\right]V_{i+1}^n = \mu\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right)V_{i-1}^k\right. \\
&\left.+ \left(1 - \frac{2\beta_2}{h^2}\right)V_i^k + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right)V_{i+1}^k\right] + \mu\gamma_{n-1}^\alpha\left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right)V_{i-1}^0 + \left(1 - \frac{2\beta_2}{h^2}\right)V_i^0\right. \\
&\left.+ \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right)V_{i+1}^0\right] + \left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right)f_{i-1}^n + \left(1 - \frac{2\beta_2}{h^2}\right)f_i^n + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right)f_{i+1}^n\right] + \hat{R}_i^n, \\
&\qquad\qquad 1 \leq i \leq M-1;\ 1 \leq n \leq N.
\end{aligned} \tag{20}$$

There exists a positive constant C_3 such that

$$\left|\hat{R}_i^n\right| \leq C_3\left(\Delta\tau^{4-\alpha} + h^4\right),\ 1 \leq i \leq M-1;\ 3 \leq n \leq N. \tag{21}$$

In (20), denoting v_i^n as the approximation of V_i^n, omitting the term $\hat{R}_i^n$ and after rearranging the terms, the $L1$-2-3-CES for the problem (12) with initial and boundary conditions is derived as follows:

$$\left[(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} + \frac{b}{2h}\right)\right] v_{i-1}^n + \left[(\mu\gamma_o^\alpha + c)\left(1 - \frac{2\beta_2}{h^2}\right) - \frac{2\beta}{h^2}\right] v_i^n$$
$$+ \left[(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} - \frac{b}{2h}\right)\right] v_{i+1}^n = \mu\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right) v_{i-1}^k\right.$$
$$\left. + \left(1 - \frac{2\beta_2}{h^2}\right) v_i^k + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right) v_{i+1}^k\right] + \mu\gamma_{n-1}^\alpha\left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right) v_{i-1}^0 + \left(1 - \frac{2\beta_2}{h^2}\right) v_i^0\right.$$
$$\left. + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right) v_{i+1}^0\right] + \left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right) f_{i-1}^n + \left(1 - \frac{2\beta_2}{h^2}\right) f_i^n + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right) f_{i+1}^n\right],$$
$$1 \le i \le M-1;\ 1 \le n \le N, \tag{22}$$

$v_i^0 = z(x_i),\ i = 1, 2, \ldots, M-1,$

$v_0^n = p(t_n),\ v_M^n = q(t_n),\ n = 0, 1, 2, \ldots, N,$

where $\mu = \dfrac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}$.

Theorem 3.4. *The $L1$-2-3-CES (22) has a unique solution.*

Proof. The scheme (22) can be expressed in the matrix form as

$$Av^n = \mathbf{b},$$

where the right-hand side $\mathbf{b}$ depends on $v^{n-1}, v^{n-2}, \ldots, v^0$ and $A = \text{tridiag}(L, D, U)$, with

$$L = \left[(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} + \frac{b}{2h}\right)\right],\ D = \left[(\mu\gamma_o^\alpha + c)\left(1 - \frac{2\beta_2}{h^2}\right) - \frac{2\beta}{h^2}\right] \text{ and}$$
$$U = \left[(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} - \frac{b}{2h}\right)\right].$$

Note that

$$\frac{\beta}{h^2} \pm \frac{b}{2h} = \frac{a}{h^2}\left(\frac{bh}{2a}\coth\left(-\frac{bh}{2a}\right) \pm \frac{bh}{2a}\right) = \frac{a}{h^2}\frac{bh}{2a}\left(\coth\left(-\frac{bh}{2a}\right) \pm 1\right).$$

If we let $z = \dfrac{bh}{2a}$, then

$$\frac{bh}{2a}\left(\coth\left(-\frac{bh}{2a}\right) \pm 1\right) = 2z\frac{e^{\mp z}}{e^{-z} - e^{z}} = 2e^{z \mp z}\frac{z}{1 - e^{2z}} \le 0, \quad z \in (-\infty, +\infty).$$

Hence

$$\left|\frac{\beta}{h^2} + \frac{b}{2h}\right| + \left|\frac{\beta}{h^2} - \frac{b}{2h}\right| = -\frac{2\beta}{h^2}.$$

Using [23, Lemma 3], we get

$$\left(1 - \frac{2\beta_2}{h^2}\right) = \left|1 - \frac{2\beta_2}{h^2}\right| > \left|\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right| + \left|\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right|.$$

Therefore, we obtain

$$\begin{aligned}|L|+|U| &= \left|(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} + \frac{b}{2h}\right)\right| + \left|(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} - \frac{b}{2h}\right)\right| \\ &\leq (\mu\gamma_o^\alpha + c)\left(\left|\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right| + \left|\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right|\right) + \left(\left|\frac{\beta}{h^2} + \frac{b}{2h}\right| + \left|\frac{\beta}{h^2} - \frac{b}{2h}\right|\right) \\ &< (\mu\gamma_o^\alpha + c)\left(1 - \frac{2\beta_2}{h^2}\right) - \frac{2\beta}{h^2} = |D|.\end{aligned}$$

This implies that the matrix A is strictly diagonally dominant. Therefore, A is nonsingular. Hence, there exists a unique solution. □

4. Stability and convergence analysis

In this section, we present a comprehensive stability and convergence analysis of the designed schemes L1-2-3-CES and L1-2-3-TCS.

Lemma 4.1. *Gronwall's lemma (Discrete version): Let α be a nonnegative constant, and let (u_n) and (w_n) be nonnegative sequences, then the following holds,*

$$\text{if } u_n \leq \alpha + \sum_{k=0}^{n-1} u_k w_k,\ \forall n,\ \text{then } u_n \leq \alpha \exp\left(\sum_{k=0}^{n-1} w_k\right),\ \forall n.$$

4.1. Stability analysis of L1-2-3-CES.

In this section, we establish the stability property of the L1-2-3-CES scheme (20) using Fourier analysis. Firstly, we denote

$$r_1 = \left[(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} + \frac{b}{2h}\right)\right],\quad r_2 = \left[(\mu\gamma_o^\alpha + c)\left(1 - \frac{2\beta_2}{h^2}\right) - \frac{2\beta}{h^2}\right],$$
$$r_3 = \left[(\mu\gamma_o^\alpha + c)\left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right) + \left(\frac{\beta}{h^2} - \frac{b}{2h}\right)\right],\quad s_1 = \left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right),$$
$$s_2 = \left(1 - \frac{2\beta_2}{h^2}\right),\quad s_3 = \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right).$$

Let $\hat{v}_i^n$ be an approximate solution of (20) and define

$$\rho_i^n = v_i^n - \hat{v}_i^n,\ 0 \leq i \leq M;\ 0 \leq n \leq N.$$

Then, the error equation of (20) is

$$\begin{aligned}&r_1\rho_{i-1}^n + r_2\rho_i^n + r_3\rho_{i+1}^n \\ &= \mu\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\left(s_1\rho_{i-1}^k + s_2\rho_i^k + s_3\rho_{i+1}^k\right) + \mu\gamma_{n-1}^\alpha\left(s_1\rho_{i-1}^0 + s_2\rho_i^0 + s_3\rho_{i+1}^0\right),\end{aligned} \tag{23}$$

and $\rho_0^n = \rho_M^n = 0$.

Now we define the grid function

$$\rho^n(x) := \begin{cases}\rho_i^n, & x_i - \frac{h}{2} < x \leq x_i + \frac{h}{2}, \\ 0, & b_d \leq x \leq b_d + \frac{h}{2} \text{ or } b_u - \frac{h}{2} < x \leq b_u,\end{cases}$$

which can be expanded as a Fourier series

$$\rho^n(x) = \sum_{l=-\infty}^{+\infty} d_n(l)\exp(\mathrm{i}2\pi lx/L);\ n = 1, 2, \dots, N;\ d_n(l) = \frac{1}{L}\int_0^L \rho^n(x)\exp(-\mathrm{i}2\pi lx/L)dx,$$

where $L = b_u - b_d$ and $\mathrm{i}^2 = -1$. Let $\rho^n = \left(\rho_1^n, \rho_2^n, \ldots, \rho_{M-1}^n\right) \in \mathbb{C}^{M-1}$, with the following norm

$$\|\rho^n\|_2 = \left(\sum_{i=1}^{M-1} h \left|\rho_i^n\right|^2\right)^{1/2} = \left[\int_0^L \left|\rho^n(x)\right|^2 dx\right]^{1/2}.$$

The application of the Parseval identity leads to $\|\rho^n\|_2^2 = \sum_{l=-\infty}^{+\infty} \left|d_n(l)\right|^2$. We suppose that the solution of (23) has the form $\rho_i^n = d_n \exp(\mathrm{i}\sigma i h)$, $\sigma = \dfrac{2\pi l}{L}$. Substituting ρ_i^n into (23), we obtain

$$d_n = \left(\frac{\mu\left[(s_1+s_3)\cos\sigma h + s_2 - \mathrm{i}\,(s_1-s_3)\sin\sigma h\right]}{(r_1+r_3)\cos\sigma h + r_2 - \mathrm{i}\,(r_1-r_3)\sin\sigma h}\right)\left(\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^{\alpha} - \gamma_{n-k}^{\alpha}\right) d_k + \gamma_{n-1}^{\alpha} d_0\right). \tag{24}$$

Lemma 4.2. *The following estimate holds*

$$\left|\frac{\mu\left[(s_1+s_3)\cos\sigma h + s_2 - \mathrm{i}\,(s_1-s_3)\sin\sigma h\right]}{(r_1+r_3)\cos\sigma h + r_2 - \mathrm{i}\,(r_1-r_3)\sin\sigma h}\right| \leq 1. \tag{25}$$

Proof. We have

$$\begin{aligned}
&\left|\frac{\mu\left[(s_1+s_3)\cos\sigma h + s_2 - \mathrm{i}\,(s_1-s_3)\sin\sigma h\right]}{(r_1+r_3)\cos\sigma h + r_2 - \mathrm{i}\,(r_1-r_3)\sin\sigma h}\right| \leq 1\\
\Leftrightarrow &\left|\mu\left[(s_1+s_3)\cos\sigma h + s_2 - \mathrm{i}\,(s_1-s_3)\sin\sigma h\right]\right| \leq \left|(r_1+r_3)\cos\sigma h + r_2 - \mathrm{i}\,(r_1-r_3)\sin\sigma h\right|\\
\Leftrightarrow &\left|\mu\left[\frac{2\beta_2}{h^2}\cos\sigma h + 1 - \frac{2\beta_2}{h^2} + \mathrm{i}\left(\frac{\beta_1}{h}\right)\sin\sigma h\right]\right|\\
&\leq \left|\left[(c+\mu\gamma_0^{\alpha})\frac{2\beta_2}{h^2} + \frac{2\beta}{h^2}\right]\cos\sigma h + (c+\mu\gamma_0^{\alpha})\left(1 - \frac{2\beta_2}{h^2}\right) - \frac{2\beta}{h^2}\right.\\
&\qquad\left. + \mathrm{i}\left[(c+\mu\gamma_0^{\alpha})\left(\frac{\beta_1}{h}\right) - \frac{b}{h}\right]\sin\sigma h\right|\\
\Leftrightarrow &\mu^2\left[\left(1 - \frac{2\beta_2}{h^2}(1-\cos\sigma h)\right)^2 + \frac{\beta_1^2}{h^2}\sin^2\sigma h\right]\\
&\leq \left[(c+\mu\gamma_0^{\alpha})\left(1 - \frac{2\beta_2}{h^2}(1-\cos\sigma h)\right) + \frac{2\beta}{h^2}(\cos\sigma h - 1)\right]^2\\
&\qquad + \left[(c+\mu\gamma_0^{\alpha})\left(\frac{\beta_1}{h}\right) - \frac{b}{h}\right]^2\sin^2\sigma h
\end{aligned}$$

$$\Leftrightarrow \left(c^2+2c\mu\gamma_0^\alpha\right)\left(1-\frac{2\beta_2}{h^2}(1-\cos\sigma h)\right)^2+\frac{4\beta^2}{h^4}(\cos\sigma h-1)^2$$
$$+2(c+\mu\gamma_0^\alpha)\left(1-\frac{2\beta_2}{h^2}(1-\cos\sigma h)\right)\frac{2\beta}{h^2}(\cos\sigma h-1)+\left(c^2+2c\mu\gamma_0^\alpha\right)\frac{\beta_1^2}{h^2}\sin^2\sigma h$$
$$-2(c+\mu\gamma_0^\alpha)\left(\frac{\beta_1}{h}\right)\frac{b}{h}\sin^2\sigma h+\frac{b^2}{h^2}\sin^2\sigma h+\mu^2\left(1-\frac{2\beta_2}{h^2}(1-\cos\sigma h)\right)^2\left((\gamma_0^\alpha)^2-1\right)$$
$$+\mu^2\frac{\beta_1^2}{h^2}\sin^2\sigma h\left((\gamma_0^\alpha)^2-1\right)\geq 0. \tag{26}$$

Since $c>0,\ \mu>0$, and $\gamma_0^\alpha\geq 1$, we have

$$\left(c^2+2c\mu\gamma_0^\alpha\right)\left(1-\frac{2\beta_2}{h^2}(1-\cos\sigma h)\right)^2\geq 0,\ \left(c^2+2c\mu\gamma_0^\alpha\right)\frac{\beta_1^2}{h^2}\sin^2\sigma h\geq 0,$$
$$\mu^2\frac{\beta_1^2}{h^2}\sin^2\sigma h\left((\gamma_0^\alpha)^2-1\right)\geq 0,\ \mu^2\left(1-\frac{2\beta_2}{h^2}(1-\cos\sigma h)\right)^2\left((\gamma_0^\alpha)^2-1\right)\geq 0.$$

Therefore, if we can prove

$$2(c+\mu\gamma_0^\alpha)\left(1-\frac{2\beta_2}{h^2}(1-\cos\sigma h)\right)\frac{2\beta}{h^2}(\cos\sigma h-1)-2(c+\mu\gamma_0^\alpha)\left(\frac{\beta_1}{h}\right)\frac{b}{h}\sin^2\sigma h\geq 0,$$

or equivalently

$$\frac{8(c+\mu\gamma_0^\alpha)}{h^2}\sin^2\frac{\sigma h}{2}\left[-\beta\left(1-\frac{4\beta_2}{h^2}\sin^2\frac{\sigma h}{2}\right)+(-b\beta_1)\cos^2\frac{\sigma h}{2}\right]\geq 0, \tag{27}$$

then (26) holds. Next, if $\sin^2\frac{\sigma h}{2}=0$, then (27) holds. When $\sin^2\frac{\sigma h}{2}\neq 0$, we have

$$-\beta\left(1-\frac{4\beta_2}{h^2}\sin^2\frac{\sigma h}{2}\right)=a(-z)\coth(-z)\left[1-\left(\frac{4a^2}{b^2h^2}(1+z\coth(-z))+\frac{2}{3}\right)\sin^2\frac{\sigma h}{2}\right]$$
$$=a(-z)\coth(-z)\left[1+\frac{4a^2}{b^2h^2}(-1+(-z)\coth(-z))\sin^2\frac{\sigma h}{2}-\frac{2}{3}\sin^2\frac{\sigma h}{2}\right],$$

and

$$(-b\beta_1)\cos^2\frac{\sigma h}{2}=a(1+z\coth(-z))\cos^2\frac{\sigma h}{2}=a\cos^2\frac{\sigma h}{2}-a(-z)\coth(-z)\cos^2\frac{\sigma h}{2},$$

where $z=\frac{bh}{2a}$. Now, we have

$$-\beta\left(1-\frac{4\beta_2}{h^2}\sin^2\frac{\sigma h}{2}\right)+(-b\beta_1)\cos^2\frac{\sigma h}{2}$$
$$=-az\coth(-z)\left[\frac{4a^2}{b^2h^2}(-1+(-z)\coth(-z))\sin^2\frac{\sigma h}{2}+\frac{1}{3}\sin^2\frac{\sigma h}{2}\right]+a\cos^2\frac{\sigma h}{2}.$$

Using [23, Lemma 1], we get

$$1-y\coth y\leq 0,\ y\in(-\infty,+\infty).$$

Therefore

$$a(-z)\coth(-z)\left[\frac{4a^2}{b^2h^2}(-1+(-z)\coth(-z))\sin^2\frac{\sigma h}{2}+\frac{1}{3}\sin^2\frac{\sigma h}{2}\right]+a\cos^2\frac{\sigma h}{2}\geq 0.$$

Then (27) holds, that is, (26) holds, which completes the proof. □

Lemma 4.3. *Suppose that d_n is the solution of (24), then there exists a positive constant K such that*

$$|d_n|\leq K\,|d_0|\,,\ n=1,2,\dots,N.$$

Proof. Using (25) in (24), we obtain

$$\begin{aligned}|d_n| &\leq \left|\left(\sum_{k=1}^{n-1}\left(\gamma^{\alpha}_{n-k-1}-\gamma^{\alpha}_{n-k}\right)d_k+\gamma^{\alpha}_{n-1}d_0\right)\right|\\ &\leq \sum_{k=1}^{n-1}\left|\left(\gamma^{\alpha}_{n-k-1}-\gamma^{\alpha}_{n-k}\right)\right||d_k|+\left|\gamma^{\alpha}_{n-1}\right||d_0|\,.\end{aligned}$$

By using Lemma 3.1, we obtain

$$|d_n|\leq\sum_{k=1}^{n-1}\left|\left(\gamma^{\alpha}_{n-k-1}-\gamma^{\alpha}_{n-k}\right)\right||d_k|+\frac{11}{6}\,|d_0|\,.$$

Based on Lemma 4.1, we have

$$|d_n|\leq\frac{11}{6}\,|d_0|\exp\left(\sum_{k=1}^{n-1}\left|\gamma^{\alpha}_{n-k-1}-\gamma^{\alpha}_{n-k}\right|\right),$$

Utilizing the triangle inequality and the properties (ii), (iii), (iv) and (v) in Lemma 3.1, we obtain

$$|d_n|\leq\frac{11}{6}\exp\left(5\gamma^{\alpha}_0\right)|d_0|\leq K\,|d_0|\,,$$

and the proof is completed. □

Theorem 4.1. *The proposed scheme $L1$-2-3-CES (20) is unconditionally stable.*

Proof. Based on Lemma 4.3 and Parseval equality, we obtain

$$\begin{aligned}\|\rho^n\|_2^2 &= \sum_{i=1}^{M-1}h\,|\rho_i^n|^2=h\sum_{i=1}^{M-1}\left|d_ne^{\mathrm{i}\sigma ih}\right|^2=h\sum_{i=1}^{M-1}|d_n|^2\\ &\leq K^2h\sum_{i=1}^{M-1}|d_0|^2=K^2h\sum_{i=1}^{M-1}\left|d_0e^{\mathrm{i}\sigma ih}\right|^2=K^2\left\|\rho^0\right\|_2^2.\end{aligned}$$

This completes the proof. □

4.2. **Stability analysis of $L1$-2-3-TCS.** In this section, we prove the stability property of the $L1$-2-3-TCS scheme (16) using Fourier analysis. Let $\hat{v}_i^n$ be an approximate solution of (16) and define

$$\epsilon_i^k=v_i^n-\hat{v}_i^n,\ 0\leq i\leq M;\ 0\leq n\leq N.$$

Then, the error equation of (16) is

$$\begin{aligned}
&\left[\mu_1 - \frac{\mu\gamma_0^\alpha + c}{12}. - \frac{b}{2h} + \frac{bh}{24a}(\mu\gamma_0^\alpha + c)\right]\epsilon_{i-1}^n + \left[-2\mu_1 - \frac{10}{12}(\mu\gamma_0^\alpha + c)\right]\epsilon_i^n \\
&+ \left[\mu_1 - \frac{\mu\gamma_0^\alpha + c}{12} + \frac{b}{2h} - \frac{bh}{24a}(\mu\gamma_0^\alpha + c)\right]\epsilon_{i+1}^n \\
&= -\mu\left(\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\epsilon_i^k + \gamma_{n-1}^\alpha\epsilon_i^0\right) \\
&- \frac{\mu}{12}\left[\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\left(\epsilon_{i-1}^k - 2\epsilon_i^k + \epsilon_{i+1}^k\right) + \gamma_{n-1}^\alpha\left(\epsilon_{i-1}^0 - 2\epsilon_i^0 + \epsilon_{i+1}^0\right)\right] \\
&- \frac{b\mu h}{24a}\left[\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\left(\epsilon_{i+1}^k - \epsilon_{i-1}^k\right) + \gamma_{n-1}^\alpha\left(\epsilon_{i+1}^0 - \epsilon_{i-1}^0\right)\right],
\end{aligned} \tag{28}$$

and $\epsilon_0^n = \epsilon_M^n = 0$.

Now we define the grid function

$$\epsilon^n(x) := \begin{cases} \epsilon_i^n, & x_i - \frac{h}{2} < x \le x_i + \frac{h}{2}, \\ 0, & b_d \le x \le b_d + \frac{h}{2} \text{ or } b_u - \frac{h}{2} < x \le b_u, \end{cases}$$

which can be expanded as a Fourier series

$$\epsilon^n(x) = \sum_{l=-\infty}^{+\infty}\hat{d}_n(l)\exp(\mathrm{i}2\pi lx/L);\ n = 1, 2, \ldots, N;\ \hat{d}_n(l) = \frac{1}{L}\int_0^L \epsilon^n(x)\exp(-\mathrm{i}2\pi lx/L)dx,$$

where, $L = b_u - b_d$ and $\mathrm{i}^2 = -1$. Let $\epsilon^n = \left(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_{M-1}^n\right) \in \mathbb{C}^{M-1}$ with the following norm

$$\|\epsilon^n\|_2 = \left(\sum_{i=1}^{M-1} h\left|\epsilon_i^n\right|^2\right)^{1/2} = \left[\int_0^L \left|\epsilon^n(x)\right|^2 dx\right]^{1/2}.$$

Then, the application of the Parseval identity leads to $\|\epsilon^n\|_2^2 = \sum_{l=-\infty}^{+\infty}\left|\hat{d}_n(l)\right|^2$. We suppose the solution of (28) has the form $\epsilon_i^n = \hat{d}_n\exp(\mathrm{i}\sigma ih)$, $\sigma = \frac{2\pi l}{L}$. Substituting ϵ_i^n into (28), we arrive at

$$\begin{aligned}
\hat{d}_n = &\left(\frac{-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\mathrm{i}}{12a}\sin(\sigma h)}{\left(-4\mu_1 + \frac{\mu\gamma_0^\alpha + c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\mathrm{i}}{h} - \frac{bh\mathrm{i}}{12a}(\mu\gamma_0^\alpha + c)\right)\sin(\sigma h) - (\mu\gamma_0^\alpha + c)}\right) \\
&\times\left(\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\hat{d}_k + \gamma_{n-1}^\alpha\hat{d}_0\right).
\end{aligned} \tag{29}$$

Lemma 4.4. *The following estimate holds*

$$\left|\frac{-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\mathrm{i}}{12a}\sin(\sigma h)}{\left(-4\mu_1 + \frac{\mu\gamma_0^\alpha + c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\mathrm{i}}{h} - \frac{bh\mathrm{i}}{12a}(\mu\gamma_0^\alpha + c)\right)\sin(\sigma h) - (\mu\gamma_0^\alpha + c)}\right| \le 1, \tag{30}$$

when $\mu_1 > \dfrac{b^2}{a}$.

Proof. Since $\gamma_0^\alpha \geq 1$ and $0 < \alpha < 1$, it follows that $\mu > 0$. By choosing h sufficiently small such that $\mu_1 > \dfrac{b^2}{a}$, the inequality in (30) holds if and only if

$$\left| -\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\mathrm{i}}{12a}\sin(\sigma h)\right| \leq \left| \left(-4\mu_1 + \frac{\mu\gamma_0^\alpha + c}{3}\right)\sin^2\frac{\sigma h}{2} \right.$$
$$\left. + \left(\frac{b\mathrm{i}}{h} - \frac{bh\mathrm{i}}{12a}(\mu\gamma_0^\alpha + c)\right)\sin(\sigma h) - (\mu\gamma_0^\alpha + c)\right|.$$

Since $c > 0$, $\mu > 0$ and $\gamma_0^\alpha \geq 1$, we have

$$\left(-\mu + \frac{\mu}{3}S\right)^2\left((\gamma_0^\alpha)^2 - 1\right) \geq 0, \quad \frac{b^2h^2\mu^2}{12^2a^2}(\sin\sigma h)^2\left((\gamma_0^\alpha)^2 - 1\right) \geq 0, \quad \left(-c + \frac{c}{3}S\right)^2 \geq 0,$$

$$\frac{b^2h^2\mu\gamma_0^\alpha c}{12^2a^2}(\sin\sigma h)^2 \geq 0, \quad \left(-\mu\gamma_0^\alpha + \frac{\mu\gamma_0^\alpha S}{3}\right)\left(-c + \frac{cS}{3}\right) \geq 0, \quad \frac{b^2h^2c^2}{12^2a^2}(\sin\sigma h)^2 \geq 0,$$

where $S = \sin^2\left(\dfrac{\sigma h}{2}\right)$. Thus, we have to prove

$$(-4\mu_1 S)^2 - 8\mu_1 S\left(-\mu\gamma_0^\alpha + \frac{\mu\gamma_0^\alpha}{3}S\right) - 8\mu_1 S\left(-c + \frac{c}{3}S\right)$$
$$+ \left(-\frac{b^2}{12a}\mu\gamma_0^\alpha - \frac{b^2}{12a}c + \frac{b^2}{h^2}\right)\sin^2(\sigma h) \geq 0. \tag{31}$$

Since

$$-\frac{b^2}{12a}\mu\gamma_0^\alpha\sin^2(\sigma h) = \frac{2b^2}{6a}\mu\gamma_0^\alpha S^2 - \frac{2b^2}{6a}\mu\gamma_0^\alpha S \quad \text{and} \quad -\frac{b^2}{12a}c\sin^2(\sigma h) = \frac{2b^2}{6a}cS^2 - \frac{2b^2}{6a}cS,$$

the inequality (31) holds if and only if

$$(4\mu_1 S)^2 + 8\mu_1\gamma_0^\alpha S\left(\mu - \frac{\mu}{3}S\right) + (8\mu_1 S)\left(c - \frac{c}{3}S\right) + \left(\frac{2b^2}{6a}\mu\gamma_0^\alpha S^2 - \frac{2b^2}{6a}\mu\gamma_0^\alpha S\right)$$
$$+ \left(\frac{2b^2}{6a}cS^2 - \frac{2b^2}{6a}cS\right) + \frac{b^2}{h^2}\sin^2(\sigma h) \geq 0,$$

or equivalently,

$$(4\mu_1 S)^2 + \left(\frac{8}{3}\mu\gamma_0^\alpha\mu_1 S - \frac{8}{3}\mu\gamma_0^\alpha\mu_1 S^2\right) + \left(\frac{2}{6}\mu\gamma_0^\alpha\mu_1 S - \frac{2b^2}{6a}\mu\gamma_0^\alpha S\right) + \left(\frac{8}{3}c\mu_1 S - \frac{8}{3}c\mu_1 S^2\right)$$
$$+ \frac{2b^2}{6a}\mu\gamma_0^\alpha S^2 + \frac{30}{6}\mu\gamma_0^\alpha\mu_1 S + \left(\frac{2}{6}\mu_1 cS - \frac{2b^2}{6a}cS\right) + \frac{2b^2}{6a}cS^2 + \frac{30}{6}\mu_1 cS + \frac{b^2}{h^2}\sin^2(\sigma h) \geq 0. \tag{32}$$

Since

$$\left(\frac{8}{3}\mu\gamma_0^\alpha\mu_1 S - \frac{8}{3}\mu\gamma_0^\alpha\mu_1 S^2\right) = \frac{8}{3}\mu\gamma_0^\alpha\mu_1 S\cos^2\frac{\sigma h}{2}, \quad \left(\frac{8}{3}c\mu_1 S - \frac{8}{3}c\mu_1 S^2\right) = \frac{8}{3}c\mu_1 S\cos^2\frac{\sigma h}{2},$$

and

$$\left(\frac{2}{6}\mu\gamma_0^\alpha\mu_1 S - \frac{2b^2}{6a}\mu\gamma_0^\alpha S\right) = \frac{2}{6}\mu\gamma_0^\alpha S\left(\mu_1 - \frac{b^2}{a}\right), \quad \left(\frac{2}{6}\mu_1 cS - \frac{2b^2}{6a}cS\right) = \frac{2}{6}cS\left(\mu_1 - \frac{b^2}{a}\right).$$

All terms of (32) are positive, so (30) holds, which completes the proof. □

Lemma 4.5. *Suppose that $\hat{d}_n$ is the solution of (29), then there exists a positive constant $\hat{K}$ such that*

$$\left|\hat{d}_n\right| \leq \hat{K}\left|\hat{d}_0\right|;\ n = 1, 2, \ldots, N.$$

Proof. Using (30) in (29), we obtain

$$\begin{aligned}|\hat{d}_n| &\leq \left|\left(\sum_{k=1}^{n-1}\left(\gamma^{\alpha}_{n-k-1} - \gamma^{\alpha}_{n-k}\right)\hat{d}_k + \gamma^{\alpha}_{n-1}\hat{d}_0\right)\right| \\ &\leq \sum_{k=1}^{n-1}\left|\left(\gamma^{\alpha}_{n-k-1} - \gamma^{\alpha}_{n-k}\right)\right|\left|\hat{d}_k\right| + \left|\gamma^{\alpha}_{n-1}\right|\left|\hat{d}_0\right|.\end{aligned}$$

Using Lemma 3.1, we obtain

$$|\hat{d}_n| \leq \sum_{k=1}^{n-1}\left|\left(\gamma^{\alpha}_{n-k-1} - \gamma^{\alpha}_{n-k}\right)\right|\left|\hat{d}_k\right| + \frac{11}{6}\left|\hat{d}_0\right|.$$

Based on Lemma 4.1, we have

$$\left|\hat{d}_n\right| \leq \frac{11}{6}\left|\hat{d}_0\right|\exp\left(\sum_{k=1}^{n-1}\left|\gamma^{\alpha}_{n-k-1} - \gamma^{\alpha}_{n-k}\right|\right),$$

Utilizing the triangle inequality and the properties (ii), (iii), (iv) and (v) in Lemma 3.1, we obtain

$$\left|\hat{d}_n\right| \leq \frac{11}{6}\exp\left(5\gamma^{\alpha}_0\right)\left|\hat{d}_0\right| \leq \hat{K}\left|\hat{d}_0\right|,$$

and the proof is completed. □

Theorem 4.2. *The $L1$-2-3-TCS (16) is stable.*

Proof. Based on Lemma 4.5 and Parseval equality, we obtain

$$\begin{aligned}\|\epsilon^n\|_2^2 &= \sum_{i=1}^{M-1} h\left|\epsilon^n_i\right|^2 = h\sum_{i=1}^{M-1}\left|\hat{d}_n e^{\mathrm{i}\sigma ih}\right|^2 = h\sum_{i=1}^{M-1}\left|\hat{d}_n\right|^2 \\ &\leq \hat{K}^2 h\sum_{i=1}^{M-1}\left|\hat{d}_0\right|^2 = \hat{K}^2 h\sum_{i=1}^{M-1}\left|\hat{d}_0 e^{\mathrm{i}\sigma ih}\right|^2 = \hat{K}^2\left\|\epsilon^0\right\|_2^2.\end{aligned}$$

So, the $L1$-2-3-TCS (16) is stable. □

4.3. **Convergence analysis of $L1$-2-3-CES.** In this section, the convergence analysis of the $L1$-2-3-CES numerical technique is studied using the Fourier analysis approach. Let $e^n_i = V^n_i - v^n_i,\ 1 \leq i \leq M-1,\ 1 \leq n \leq N$. Subtracting (22) from (20), we obtain

$$r_1 e^n_{i-1} + r_2 e^n_i + r_3 e^n_{i+1} = \mu\sum_{k=1}^{n-1}\left(\gamma^{\alpha}_{n-k-1} - \gamma^{\alpha}_{n-k}\right)\left(s_1 e^k_{i-1} + s_2 e^k_i + s_3 e^k_{i+1}\right) + \hat{R}^n_i, \quad (33)$$

where

$$e^0_i = 0,\ 1 \leq i \leq M-1,$$

$$e^n_0 = e^n_M = 0,\ 0 \leq n \leq N.$$

Define the following grid functions:

$$e^n(x)=\begin{cases} e_i^n, & x_i-\frac{h}{2}<x\le x_i+\frac{h}{2}, \quad 1\le i\le M-1,\\ 0, & b_d\le x\le b_d+\frac{h}{2} \text{ or } b_u-\frac{h}{2}<x\le b_u,\end{cases}$$

and

$$\hat{R}^n(x)=\begin{cases} \hat{R}_i^n, & x_i-\frac{h}{2}<x\le x_i+\frac{h}{2}, \quad 1\le i\le M-1,\\ 0, & b_d\le x\le b_d+\frac{h}{2} \text{ or } b_u-\frac{h}{2}<x\le b_u,\end{cases}$$

with

$$e^n=\left(e_1^n,e_2^n,\dots,e_{M-1}^n\right)^T,\quad \hat{R}^n=\left(\hat{R}_1^n,\hat{R}_2^n,\dots,\hat{R}_{M-1}^n\right)^T.$$

Thus $e^n(x)$ and $\hat{R}^n(x)$ have the following Fourier series expansions

$$e^n(x)=\sum_{l=-\infty}^{\infty}\eta_n(l)e^{2\pi\mathrm{i}lx/L},\quad \hat{R}^n(x)=\sum_{l=-\infty}^{\infty}\zeta_n(l)e^{2\pi\mathrm{i}lx/L},$$

where

$$L=b_u-b_d,\quad \eta_n(l)=\frac{1}{L}\int_0^L e^n(x)e^{-2\pi\mathrm{i}lx/L}dx,\quad \zeta_n(l)=\frac{1}{L}\int_0^L \hat{R}^n(x)e^{-2\pi\mathrm{i}lx/L}dx.$$

Using the Parseval equality and the definition of L^2 norm, we get

$$\|e^n\|_2^2=\sum_{i=1}^{M-1}h\left|e_i^n\right|^2=\int_0^L\left|e^n(x)\right|^2dx=\sum_{l=-\infty}^{\infty}\left|\eta_n(l)\right|^2, \tag{34}$$

$$\left\|\hat{R}^n\right\|_2^2=\sum_{i=1}^{M-1}h\left|\hat{R}_i^n\right|^2=\int_0^L\left|\hat{R}^n(x)\right|^2dx=\sum_{l=-\infty}^{\infty}\left|\zeta_n(l)\right|^2. \tag{35}$$

Using (21) and (35), we obtain

$$\left\|\hat{R}^n\right\|_2\le\sqrt{Mh}C_3\left(\Delta\tau^{4-\alpha}+h^4\right)=C_3\sqrt{L}\left(\Delta\tau^{4-\alpha}+h^4\right)\le C\left(\Delta\tau^{4-\alpha}+h^4\right). \tag{36}$$

We assume that the solutions of (33) are

$$e_i^n=\eta_n e^{\mathrm{i}\sigma ih},\quad \hat{R}_i^n=\zeta_n e^{\mathrm{i}\sigma ih}, \tag{37}$$

where $\sigma=2\pi l/L$.

Substituting (37) in (33), we have

$$\eta_n=\frac{\mu\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^{\alpha}-\gamma_{n-k}^{\alpha}\right)\left[(s_1+s_3)\cos\sigma h+s_2-\mathrm{i}\,(s_1-s_3)\sin\sigma h\right]\eta_k+\zeta_n}{(r_1+r_3)\cos\sigma h+r_2-\mathrm{i}\,(r_1-r_3)\sin\sigma h}. \tag{38}$$

Lemma 4.6. *For $0<\Delta\tau<1$, we have*

$$\left|(r_1+r_3)\cos\sigma h+r_2-\mathrm{i}\,(r_1-r_3)\sin\sigma h\right|\ge\frac{1}{3}.$$

Proof. Since $0<\alpha<1$ and $0<\Delta\tau\leq 1$, we have $0<\Gamma(2-\alpha)<1$. So $\mu=\dfrac{\Delta\tau^{-\alpha}}{\Gamma(2-\alpha)}>1$ and $\gamma_o^{\alpha}\geq 1$. Thus, we can write

$$\begin{aligned}
&|\,(r_1+r_3)\cos\sigma h + r_2 - \mathrm{i}\,(r_1-r_3)\sin\sigma h|^2\\
&\quad=\left[\left((c+\mu\gamma_o^{\alpha})\frac{2\beta_2}{h^2}+\frac{2\beta}{h^2}\right)\cos\sigma h+(c+\mu\gamma_o^{\alpha})\left(1-\frac{2\beta_2}{h^2}\right)-\frac{2\beta}{h^2}\right]^2\\
&\qquad+\left[(c+\mu\gamma_o^{\alpha})\left(\frac{\beta_1}{h}\right)-\frac{b}{h}\right]^2\sin^2\sigma h\\
&\quad\geq\left[(c+\mu\gamma_o^{\alpha})\left(1-\frac{4\beta_2}{h^2}\sin^2\frac{\sigma h}{2}\right)-\frac{4\beta}{h^2}\sin^2\frac{\sigma h}{2}\right]^2\\
&\quad\geq\left[(c+\mu\gamma_o^{\alpha})\left(1+\frac{4a^2}{b^2h^2}(-1+(-z)\coth(-z))\sin^2\frac{\sigma h}{2}-\frac{2}{3}\sin^2\frac{\sigma h}{2}\right)\right.\\
&\qquad\left.+\frac{4a}{h^2}(-z)\coth(-z)\sin^2\frac{\sigma h}{2}\right]^2\\
&\quad\geq\left(\frac{\mu\gamma_o^{\alpha}}{3}\right)^2\geq\frac{1}{9}.
\end{aligned}$$

That is

$$|(r_1+r_3)\cos\sigma h + r_2 - \mathrm{i}\,(r_1-r_3)\sin\sigma h|\geq\frac{1}{3}.$$

Hence, the proof is complete. □

Lemma 4.7. *Let η_n be the solution of (38). Then, there is a positive constant K, such that*

$$|\eta_n|\leq K\,|\zeta_3|\,;\ n=3,4,\ldots,N.$$

Proof. Using Lemma 4.2 and Lemma 4.6 in (38), we have

$$|\eta_n|\leq\sum_{k=1}^{n}\left|\left(\gamma_{n-k-1}^{\alpha}-\gamma_{n-k}^{\alpha}\right)\right||\eta_k|+3\,|\zeta_n|\,. \tag{39}$$

The series on the right-hand side of (35) is convergent. Hence there exists a positive constant K_n, so that

$$|\zeta_n|\leq K_n\,|\zeta_3|\,;\ n=3,4\ldots,N.$$

Let $K'=\max\{K_3,K_4,\ldots,K_N\}$, so we have

$$|\zeta_n|\leq K'\,|\zeta_3|\,;\ n=3,4,\ldots,N. \tag{40}$$

From (39) and (40), we have

$$|\eta_n|\leq\sum_{k=1}^{n-1}\left|\gamma_{n-k-1}^{\alpha}-\gamma_{n-k}^{\alpha}\right||\eta_k|+3K'\,|\zeta_3|\,.$$

Based on Lemma 4.1, we have

$$|\eta_n|\leq 3K'\,|\zeta_3|\exp\left(\sum_{k=1}^{n-1}\left|\gamma_{n-k-1}^{\alpha}-\gamma_{n-k}^{\alpha}\right|\right),$$

Utilizing the triangle inequality and the properties (ii), (iii), (iv) and (v) in Lemma 3.1, we obtain

$$|\eta_n| \leq 3K' |\zeta_3| \exp(5\gamma_0^\alpha) \leq K |\zeta_3|.$$

This completes the proof of the lemma. □

Theorem 4.3. *The L1-2-3-CES (22) is convergent in the L^2 norm, and the order of convergence is $\mathcal{O}\left(\Delta\tau^{4-\alpha}+h^4\right)$.*

Proof. Applying Lemma 4.7, and using (34) and (36), we have

$$\begin{aligned} \|e^n\|_2^2 &= \sum_{i=1}^{M-1} h\,|e_i^n|^2 \\ &= h\sum_{i=1}^{M-1}\left|\eta_n e^{\mathrm{i}\sigma ih}\right|^2 = h\sum_{i=1}^{M-1}|\eta_n|^2 \leq K^2 h \sum_{i=1}^{M-1}|\zeta_3|^2 \\ &= K^2 h \sum_{i=1}^{M-1}\left|\zeta_3 e^{\mathrm{i}\sigma ih}\right|^2 = K^2\left\|\hat{R}^3\right\|_2^2 \leq K^2 C\left(\Delta\tau^{4-\alpha}+h^4\right)^2. \end{aligned}$$

This completes the proof. □

4.4. **Convergence analysis of $L1$-2-3-TCS.** Let $\hat{e}_i^n = V_i^n - v_i^n,\ 1 \leq i \leq M-1,\ 1 \leq n \leq N$. Subtracting (16) from (14), we obtain

$$\begin{aligned} &\left[\mu_1 - \frac{\mu\gamma_0^\alpha + c}{12} - \frac{b}{2h} + \frac{bh}{24a}(\mu\gamma_0^\alpha + c)\right]\hat{e}_{i-1}^n + \left[-2\mu_1 - \frac{10}{12}(\mu\gamma_0^\alpha + c)\right]\hat{e}_i^n \\ &+ \left[\mu_1 - \frac{\mu\gamma_0^\alpha + c}{12} + \frac{b}{2h} - \frac{bh}{24a}(\mu\gamma_0^\alpha + c)\right]\hat{e}_{i+1}^n \\ &= -\mu\left[\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\hat{e}_i^k\right] - \frac{\mu}{12}\left[\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\left(\hat{e}_{i-1}^k - 2\hat{e}_i^k + \hat{e}_{i+1}^k\right)\right] \\ &- \frac{b\mu h}{24a}\left[\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\left(\hat{e}_{i+1}^k - \hat{e}_{i-1}^k\right)\right] + R_i^n, \end{aligned} \tag{41}$$

with $\hat{e}_i^0 = 0,\ \ i = 1,2,\ldots,M-1;\ \ \hat{e}_0^n = 0,\ \ \hat{e}_M^n = 0,\ \ n = 0,1,2,\ldots,N.$

Denote

$$\hat{e}^n = \left(\hat{e}_1^n, \hat{e}_2^n, \ldots, \hat{e}_{M-1}^n\right)^T, \quad R^n = \left(R_1^n, R_2^n, \ldots, R_{M-1}^n\right)^T.$$

Thus, $\hat{e}^n(x)$ and $R^n(x)$ have the following Fourier series expansions

$$\hat{e}^n(x) = \sum_{l=-\infty}^{\infty}\hat{\eta}_n(l)e^{2\pi\mathrm{i}lx/L}, \quad R^n(x) = \sum_{l=-\infty}^{\infty}\hat{\zeta}_n(l)e^{2\pi\mathrm{i}lx/L},$$

where

$$L = b_u - b_d, \quad \hat{\eta}_n(l) = \frac{1}{L}\int_0^L \hat{e}^n(x)e^{-2\pi\mathrm{i}lx/L}dx, \quad \hat{\zeta}_n(l) = \frac{1}{L}\int_0^L R^n(x)e^{-2\pi\mathrm{i}lx/L}dx.$$

Using the Parseval equality and the definition of L^2 norm, we get

$$\|\hat{e}^n\|_2^2 = \sum_{i=1}^{M-1} h\,|\hat{e}_i^n|^2 = \int_0^L |\hat{e}^n(x)|^2\,dx = \sum_{l=-\infty}^{\infty}|\hat{\eta}_n(l)|^2,$$

$$\|R^n\|_2^2 = \sum_{i=1}^{M-1} h\,|R_i^n|^2 = \int_0^L |R^n(x)|^2\,dx = \sum_{l=-\infty}^{\infty} \left|\hat{\zeta}_n(l)\right|^2.$$

Also, (15) gives

$$|R_i^n| \le C_2\left(\Delta\tau^{4-\alpha} + h^4\right), \quad 1 \le i \le M-1;\ 3 \le n \le N.$$

We assume that the solutions of (41) have the expressions

$$\hat{e}_i^n = \hat{\eta}_n e^{\mathrm{i}\sigma ih}, \quad R_i^n = \hat{\zeta}_n e^{\mathrm{i}\sigma ih}, \tag{42}$$

where $\sigma = 2\pi l/L$.

Substituting (42) in (41) gives

$$\begin{aligned}\hat{\eta}_n =& \frac{\left(-\mu + \frac{\mu}{3}\sin^2\frac{\sigma h}{2} - \frac{bh\mu\mathrm{i}}{12a}\sin(\sigma h)\right)\left(\sum_{k=1}^{n-1}\left(\gamma_{n-k-1}^\alpha - \gamma_{n-k}^\alpha\right)\hat{\eta}_k\right)}{\left(-4\mu_1 + \frac{\mu\gamma_0^\alpha + c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\mathrm{i}}{h} - \frac{bh\mathrm{i}}{12a}(\mu\gamma_0^\alpha + c)\right)\sin(\sigma h) - (\mu\gamma_0^\alpha + c)} \\ &+ \frac{\hat{\xi}_n}{\left(-4\mu_1 + \frac{\mu\gamma_0^\alpha + c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\mathrm{i}}{h} - \frac{bh\mathrm{i}}{12a}(\mu\gamma_0^\alpha + c)\right)\sin(\sigma h) - (\mu\gamma_0^\alpha + c)}.\end{aligned} \tag{43}$$

Lemma 4.8. *The following relation holds*

$$\frac{1}{\left|\left(-4\mu_1 + \frac{\mu\gamma_0^\alpha + c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\mathrm{i}}{h} - \frac{bh\mathrm{i}}{12a}(\mu\gamma_0^\alpha + c)\right)\sin(\sigma h) - (\mu\gamma_0^\alpha + c)\right|} \le 3.$$

Proof. Since $\mu,\ \gamma_o^\alpha \ge 1$ for $\Delta\tau \in (0,1)$ and $c > 0$, we obtain

$$\begin{aligned}1 &\le (2(\mu\gamma_0^\alpha + c))^2 = 9\left(\frac{2}{3}(\mu\gamma_0^\alpha + c)\right)^2 = 9\left(\mu\gamma_0^\alpha + c - \frac{\mu\gamma_0^\alpha + c}{3}\right)^2 \\ &\le 9\left(\mu\gamma_0^\alpha + c - \frac{\mu\gamma_0^\alpha + c}{3}\sin^2\frac{\sigma h}{2}\right)^2 \\ &\le 9\Bigg[\left(4\mu_1\sin^2\frac{\sigma h}{2} + \mu\gamma_0^\alpha + c - \frac{\mu\gamma_0^\alpha + c}{3}\sin^2\frac{\sigma h}{2}\right)^2 \\ &\qquad + \left(\frac{bh(\mu\gamma_0^\alpha + c)}{12a}\sin(\sigma h) - \frac{b}{h}\sin(\sigma h)\right)^2\Bigg].\end{aligned}$$

Also, we know

$$\begin{aligned}&\left|\left(-4\mu_1 + \frac{\mu\gamma_0^\alpha + c}{3}\right)\sin^2\frac{\sigma h}{2} + \left(\frac{b\mathrm{i}}{h} - \frac{bh\mathrm{i}}{12a}(\mu\gamma_0^\alpha + c)\right)\sin(\sigma h) - (\mu\gamma_0^\alpha + c)\right| \\ &= \left|\left(4\mu_1 - \frac{\mu\gamma_0^\alpha + c}{3}\right)\sin^2\frac{\sigma h}{2} - \left(\frac{b\mathrm{i}}{h} - \frac{bh\mathrm{i}}{12a}(\mu\gamma_0^\alpha + c)\right)\sin(\sigma h) + (\mu\gamma_0^\alpha + c)\right|.\end{aligned}$$

Hence, the proof is complete. □

Lemma 4.9. *Let $\hat{\eta}_n$ be the solution of (43), then there is a positive constant K, such that*

$$|\hat{\eta}_n| \le K\left|\hat{\zeta}_3\right|, \quad n = 3, 4, \ldots, N.$$

Proof. This lemma can be proved with a similar procedure as used in Lemma 4.7. □

Theorem 4.4. *The $L1$-2-3-TCS (16) is convergent in the L^2 norm, and the order of convergence is* $\mathcal{O}\left(\Delta\tau^{4-\alpha}+h^4\right)$.

Proof. This theorem can be proved with arguments similar to those used in Theorem 4.3 and is omitted here.

□

5. Numerical experiments

In this section, we present numerical experiments using the proposed schemes to verify the theoretical order of accuracy in both space and time and compare the results with the results in [6, 24]. The error is defined by $E(h,\Delta\tau)=\max\limits_{1\le i\le M-1}\left|V\left(x_i,t_N\right)-v_i^N\right|$, where $V\left(x_i,t_N\right)$ and v_i^N are the exact and approximate solutions at the grid point (x_i,t_N). The computational order of convergence in time and space are calculated using $\log_2\dfrac{E(h,\Delta\tau)}{E(h,\Delta\tau/2)}$ and $\log_2\dfrac{E(h,\Delta\tau)}{E(h/2,\Delta\tau)}$ respectively.

Example 5.1. Consider

$$\begin{cases} {}_0^C D_t^\alpha V(x,t)-\dfrac{\sigma^2}{2}\dfrac{\partial^2 V(x,t)}{\partial x^2}-\left(r-\dfrac{\sigma^2}{2}\right)\dfrac{\partial V(x,t)}{\partial x}+rV(x,t)=f(x,t), \\ \hfill 0<x<1,\ \ 0<t\le 1, \\ V(x,0)=\sin\pi x+1,\ 0<x<1, \\ V(0,t)=t^3+1,\ V(1,t)=t^3+1,\ 0\le t\le 1, \end{cases}$$

with $r=0.06$ and $\sigma=0.2$, where

$$\begin{aligned} f(x,t)=&\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}(\sin\pi x+1)-\left(t^3+1\right)\left[\frac{\sigma^2}{2}\left(-\pi^2\sin\pi x\right)+\left(r-\frac{\sigma^2}{2}\right)\pi\cos\pi x\right. \\ &\left.-r(\sin\pi x+1)\right]. \end{aligned}$$

The exact solution of the problem is $V(x,t)=\left(t^3+1\right)(\sin\pi x+1)$.

Table 1 represents the numerical errors and temporal convergence order of $L1$-2-3-Padé scheme, and the errors are compared with the errors of Padé scheme of Algorithm 1 in [24], with a fixed spatial step size $h=1/5000$ and for different $\Delta\tau$ corresponding to parameter $\alpha=0.2,0.5$ and 0.8, while Table 2 represents the same for $L1$-2-3-TCS, $L1$-2-3-CES and the results are compared with the scheme in [6]. The results reveal that the newly designed algorithms have achieved the temporal convergence order $4-\alpha$ and improved the results in [24, 6].

		Algorithm 1[24]		L1-2-3-Padé Scheme	
α	$\Delta\tau$	$E(h,\Delta\tau)$	order	$E(h,\Delta\tau)$	order
0.2	1/10	1.0741e-02	–	4.828268e-05	–
	1/20	3.4206e-03	1.6508	3.247737e-06	3.8940
	1/40	1.0646e-03	1.6840	2.251528e-07	3.8505
	1/80	3.2601e-04	1.7073	1.596850e-08	3.8176
0.5	1/10	5.0472e-02	–	4.962600e-04	–
	1/20	1.8850e-02	1.4209	4.175109e-05	3.5712
	1/40	6.9137e-03	1.4470	3.601869e-06	3.5350
	1/80	2.5057e-03	1.4642	3.144930e-07	3.5176
0.8	1/10	1.5235e-01	–	3.013250e-03	–
	1/20	6.7532e-02	1.1737	3.194413e-04	3.3277
	1/40	2.9750e-02	1.1827	3.432275e-05	3.2183
	1/80	1.3047e-02	1.1892	3.711773e-06	3.2090

TABLE 1. Numerical errors and temporal convergence orders of L1-2-3-Padé Scheme for Example 5.1 with different α, when h=1/5000.

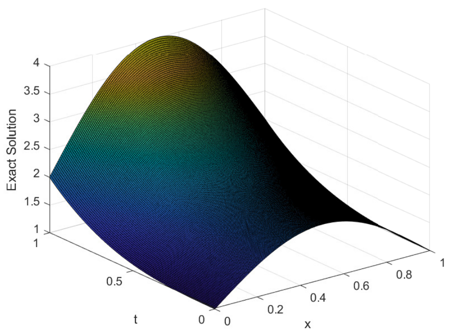
		Scheme [6]		L1-2-3-TCS		L1-2-3-CES	
α	$\Delta\tau$	$E(h,\Delta\tau)$	order	$E(h,\Delta\tau)$	order	$E(h,\Delta\tau)$	order
0.2	1/10	1.07408e-02	–	4.82827e-05	–	4.82827e-05	–
	1/20	3.42060e-03	1.6507	3.24742e-06	3.8938	3.24797e-06	3.8938
	1/40	1.06458e-03	1.6839	2.25211e-07	3.8499	2.25211e-07	3.8501
	1/80	3.26014e-04	1.7072	1.56612e-08	3.8460	1.56603e-08	3.8460
0.5	1/10	5.04722e-02	–	4.96259e-04	–	4.96260e-04	–
	1/20	1.88500e-02	1.4209	4.17508e-05	3.5712	4.17509e-05	3.5712
	1/40	6.91373e-03	1.4470	3.60158e-06	3.5351	3.60158e-06	3.5351
	1/80	2.50572e-03	1.4642	3.14559e-07	3.5172	3.14383e-07	3.5180
0.8	1/10	1.52348e-01	–	3.01325e-03	–	3.01325e-03	–
	1/20	6.75320e-02	1.1737	3.19441e-04	3.2376	3.19551e-04	3.2377
	1/40	2.97499e-02	1.1826	3.43224e-05	3.2183	3.43225e-05	3.2183
	1/80	1.30474e-02	1.1891	3.71153e-06	3.2090	3.71133e-06	3.2091

TABLE 2. Numerical errors and temporal convergence orders of L1-2-3-TCS and L1-2-3-CES schemes for Example 5.1 with different α, when h=1/5000.

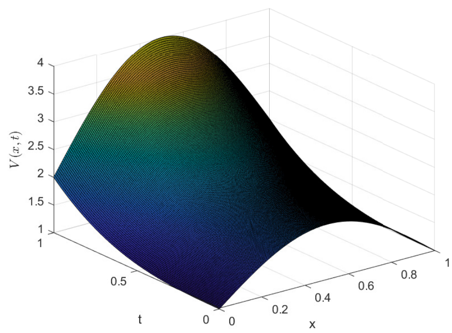
The numerical errors and spatial convergence orders of L1-2-3-Padé Scheme, L1-2-3-TCS and L1-2-3-CES with different values of α are given in Table 3 . For a fixed temporal step size $\Delta\tau = 1/10000$, the results show that L1-2-3-Padé Scheme, L1-2-3-TCS and L1-2-3-CES are convergent with the fourth order accuracy in the spatial direction.

		$L1$-2-3-Padé Scheme		$L1$-2-3-TCS		$L1$-2-3-CES	
α	h	$E(h,\Delta\tau)$	order	$E(h,\Delta\tau)$	order	$E(h,\Delta\tau)$	order
0.2	1/4	4.67455e-04	–	6.23083e-04	–	5.80789e-03	–
	1/8	2.83500e-05	4.0434	3.87131e-05	4.0085	3.61005e-05	4.0079
	1/16	1.78505e-06	3.9893	2.42019e-06	3.9996	2.25773e-06	3.9990
	1/32	1.12068e-07	3.9935	1.51716e-07	3.9957	1.42726e-07	3.9835
0.5	1/4	4.38362e-04	–	5.84684e-04	–	5.45802e-04	–
	1/8	2.65725e-05	4.0442	3.62620e-05	4.0111	3.38614e-05	4.0106
	1/16	1.67871e-06	3.9845	2.26654e-06	3.9998	2.11882e-06	3.9983
	1/32	1.05204e-07	3.9960	1.42549e-07	3.9909	1.34282e-07	3.9799
0.8	1/4	4.08798e-04	–	5.45843e-04	–	5.10640e-04	–
	1/8	2.47725e-05	4.0446	3.37933e-05	4.0137	3.16214e-05	4.0133
	1/16	1.57185e-06	3.9782	2.11183e-06	4.0002	1.99570e-06	3.9859
	1/32	9.82824e-06	3.9993	1.33410e-07	3.9846	1.25929e-07	3.9862

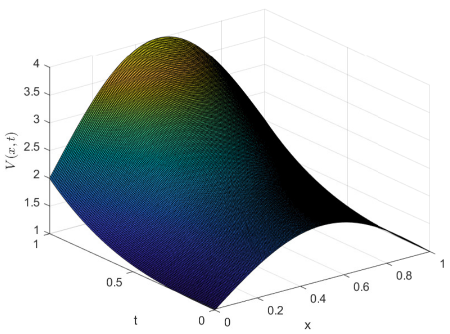
TABLE 3. Numerical errors and spatial convergence orders of $L1$-2-3-Padé Scheme, $L1$-2-3-TCS and $L1$-2-3-CES for Example 5.1 with $\Delta\tau$ =1/10000.



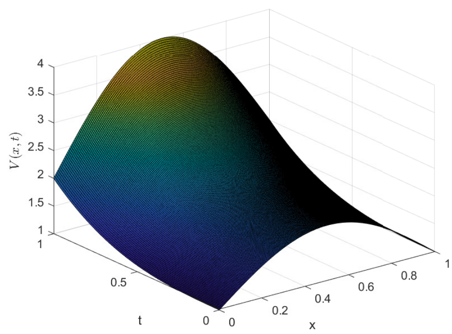
(a) Exact Solution



(b) $L1$-2-3-Padé approximate solution



(c) $L1$-2-3-TCS approximate solution



(d) $L1$-2-3-CES approximate solution

FIGURE 1. Space-time graph of the exact and numerical solution for Example 5.1 with $\alpha = 0.2$ and $M = N = 200$.

Figure 1 illustrates the space-time graph of the exact and numerical solution for Example 5.1 with $\alpha = 0.2$ and $M = N = 200$. From the figures, it is evident that the computed solutions using the proposed schemes are in good agreement with the exact solution.

Example 5.2. Consider

$$\begin{cases} {}_0^C D_t^\alpha V(x,t) - \dfrac{\sigma^2}{2}\dfrac{\partial^2 V(x,t)}{\partial x^2} - \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial V(x,t)}{\partial x} + rV(x,t) = f(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1,\ 0 < t \le 1, \\ V(x,0) = e^x,\ 0 < x < 1, \\ V(0,t) = t^3 + 1,\ V(1,t) = e\left(t^3+1\right),\ 0 \le t \le 1, \end{cases}$$

with $r = 0.06$ and $\sigma = 0.1$, where

$$f(x,t) = e^x \left(\frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} - \left(\frac{\sigma^2}{2} + \left(r - \frac{\sigma^2}{2}\right) - r \right)\left(t^3+1\right) \right).$$

The exact solution of the problem is $V(x,t) = e^x\left(t^3+1\right)$.

The numerical errors and temporal convergence orders corresponding to the proposed schemes are presented in Table 4 for Example 5.2. The computations were carried out using a fixed spatial step size of $h = 1/300$ and for three values of the fractional order parameter, $\alpha = 0.2$, 0.5 and 0.8. The numerical results confirm that all three methods successfully achieve the expected temporal convergence order of order $4-\alpha$, demonstrating the accuracy and robustness of the proposed time discretization strategies. In addition, Table 5 presents the spatial convergence orders for the proposed three numerical schemes applied to Example 5.2, using a fixed number of time steps $N = 400$ and varying spatial resolutions, for $\alpha = 0.2$, 0.5 and 0.8. The tabulated results clearly indicate that all the proposed schemes achieve fourth-order accuracy in space, thereby validating the effectiveness of the spatial discretization techniques employed in this study.

		L1-2-3-Padé Scheme		L1-2-3-TCS		L1-2-3-CES	
α	$\Delta\tau$	$E(h,\Delta\tau)$	order	$E(h,\Delta\tau)$	order	$E(h,\Delta\tau)$	order
0.2	1/16	8.9927e-06	–	8.9933e-06	–	8.9933e-06	–
	1/32	6.2148e-07	3.8550	6.2215e-07	3.8535	6.2215e-07	3.8535
	1/64	4.3130e-08	3.8489	4.3796e-08	3.8284	4.3796e-08	3.8284
	1/128	3.1652e-09	3.8146	3.1076e-09	3.8169	3.1076e-09	3.8169
0.5	1/16	1.1028e-04	–	1.1028e-04	–	1.1028e-04	–
	1/32	9.4962e-06	3.5377	9.4968e-06	3.5376	9.4968e-06	3.5376
	1/64	8.2801e-07	3.5196	8.2861e-07	3.5187	8.2861e-07	3.5187
	1/128	7.2157e-08	3.5204	7.2754e-08	3.5096	7.2754e-08	3.5096
0.8	1/16	7.7885e-04	–	7.7885e-04	–	7.7885e-04	–
	1/32	8.3626e-05	3.2193	8.3627e-05	3.2193	8.3627e-05	3.2193
	1/64	9.0402e-06	3.2095	9.0408e-06	3.2094	9.0408e-06	3.2094
	1/128	9.8006e-07	3.2054	9.8060e-07	3.2047	9.8060e-07	3.2047

TABLE 4. Numerical errors and temporal convergence orders of L1-2-3-Padé Scheme, L1-2-3-TCS and L1-2-3-CES for Example 5.2 with h =1/300.

		L1-2-3-Padé Scheme		L1-2-3-TCS		L1-2-3-CES	
α	h	$E(h, \Delta\tau)$	order	$E(h, \Delta\tau)$	order	$E(h, \Delta\tau)$	order
0.2	1/4	2.3622e-02	–	7.0302e-05	–	7.3997e-05	–
	1/8	1.3534e-03	4.1255	4.2586e-06	4.0451	4.8396e-06	3.9345
	1/16	8.5439e-05	3.9855	2.6763e-07	3.9921	3.1100e-07	3.9599
	1/32	5.3627e-06	3.9939	1.6731e-08	3.9997	1.9562e-08	3.9908
0.5	1/4	2.1771e-02	–	6.4592e-05	–	6.7993e-05	–
	1/8	1.2325e-03	4.1427	3.8776e-06	4.0581	4.4068e-06	3.9476
	1/16	7.7773e-05	3.9862	2.4240e-07	3.9997	2.8188e-07	3.9666
	1/32	4.8865e-06	3.9924	1.4102e-08	4.1034	1.6682e-08	4.0787
0.8	1/4	1.9778e-02	–	5.8385e-05	–	6.1469e-05	–
	1/8	1.0987e-03	4.1700	3.4334e-06	4.0879	3.9053e-06	3.9763
	1/16	6.9899e-05	3.9744	1.9698e-07	4.1235	2.3247e-07	4.0703
	1/32	4.3702e-06	3.9995	1.1209e-08	3.9655	1.0472e-08	4.4724

TABLE 5. Numerical errors and spatial convergence orders of L1-2-3-Padé Scheme, L1-2-3-TCS and L1-2-3-CES for Example 5.2 with $\Delta\tau$ =1/400.

Example 5.3. Consider the following time-fractional Black-Scholes equation governing European put option

$$\begin{cases} {}_0^C D_t^\alpha V(x,t) - \dfrac{\sigma^2}{2}\dfrac{\partial^2 V(x,t)}{\partial x^2} - \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial V(x,t)}{\partial x} + rV(x,t) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad -4.6 < x < 4.6,\ 0 < t \le 1, \\ V(x,0) = \max\{K - e^x, 0\},\ -4.6 < x < 4.6, \\ V(-4.6,t) = Ke^{-rt},\ V(4.6,t) = 0,\ 0 \le t \le 1. \end{cases}$$

We present the numerical results for Example 5.3, using L1-2-3-Padé scheme, L1-2-3-TCS and L1-2-3-CES, illustrating the pricing of European put options with $M = 400$ spatial grid points and $N = 80$ temporal grid points in Figures 2–5. Here K denotes the strike price. Figure 2 demonstrates the influence of the fractional order α on the option price for $\alpha = 0.2$, 0.5 and 0.8, while keeping the other parameters fixed as $K = 50$, $\sigma = 0.25$ and $r = 0.05$. The results clearly indicate that the order of the time-fractional derivative significantly affects the option pricing behavior, particularly near $\log K$. The corresponding zoomed image is presented near the domain [3.88, 3.96].

Figure 3 illustrates the impact of varying the risk-free interest rate r on the European put option price, with the parameters $K = 50$, $\sigma = 0.25$ and $\alpha = 0.2$. For values of $r = 0.005$, 0.05, 0.1 and 0.2, the numerical results show that an increase in the interest rate leads to a decrease in the put option price.

Figure 4 presents the variation in option price with respect to the strike price K, for fixed values of $\sigma = 0.25$, $r = 0.05$ and $\alpha = 0.2$. It can be observed that the option price increases monotonically with the strike price. Figure 5 represents the behaviour of the option price with respect to time for fixed values of $K = 50$, $\sigma = 0.25$, $r = 0.05$ and $\alpha = 0.2$. It is observed that as time increases, the option price also increases, which is consistent with the expectation of the real market.

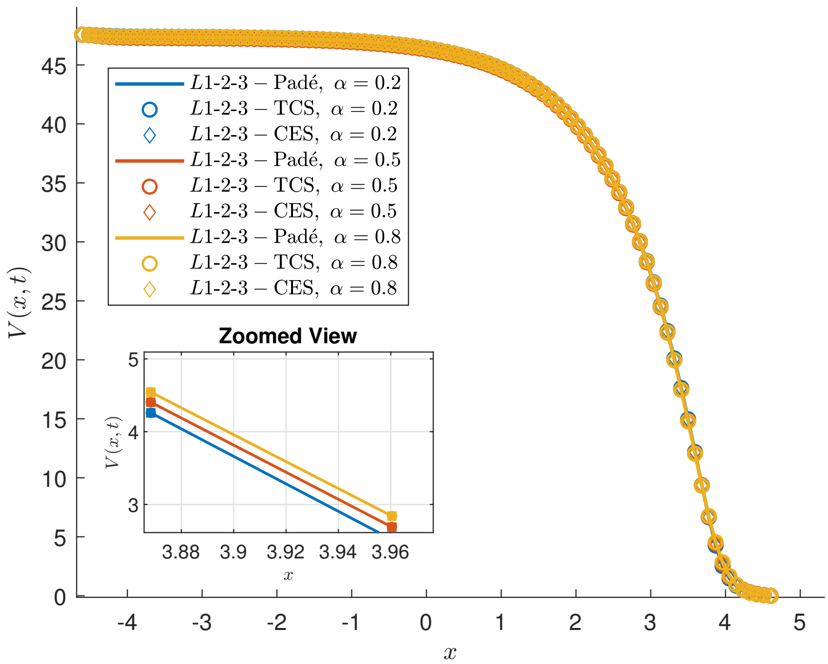


FIGURE 2. European put option price for Example 5.3 with different values of α at $t = 1$.

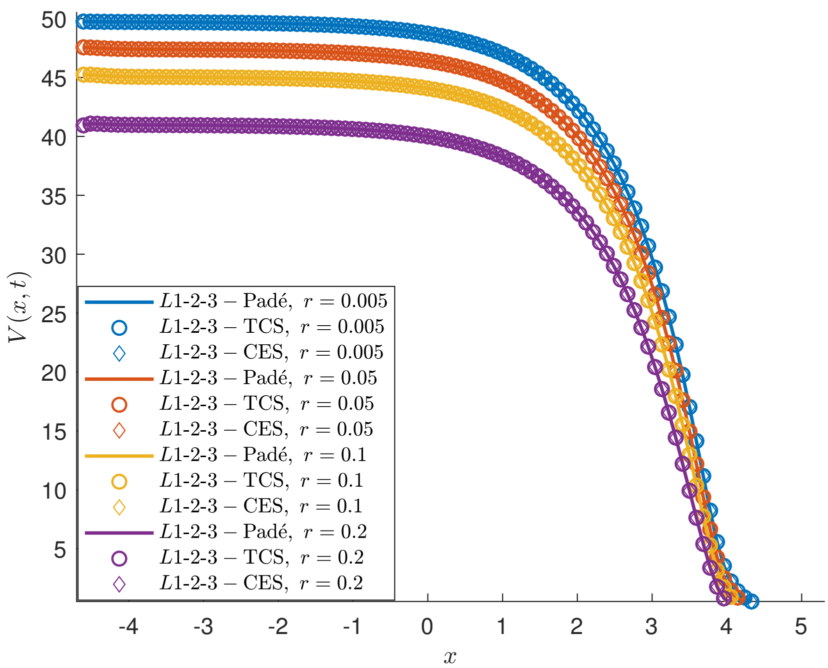


FIGURE 3. European put option price for Example 5.3 with different values of r at $t = 1$.

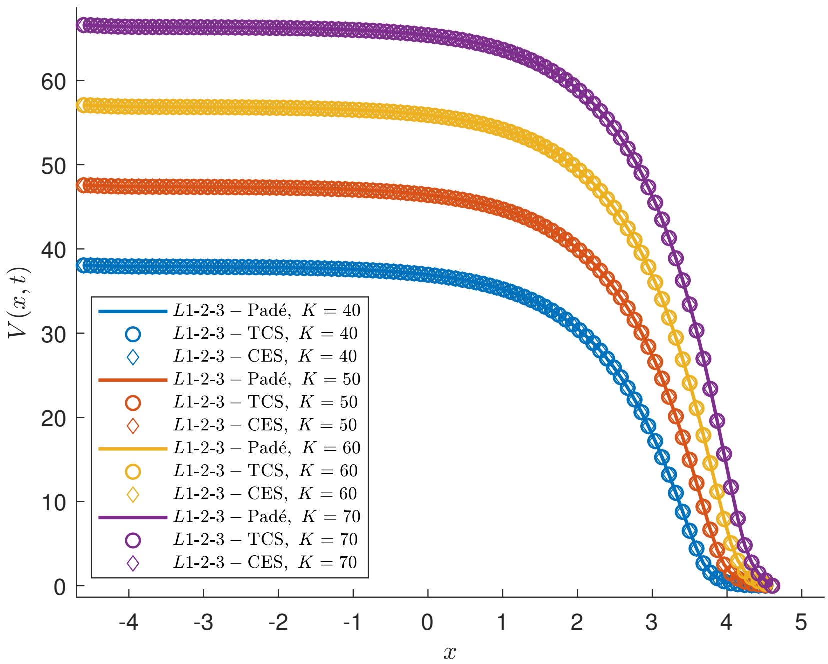


FIGURE 4. European put option price for Example 5.3 with different values of K at $t = 1$.

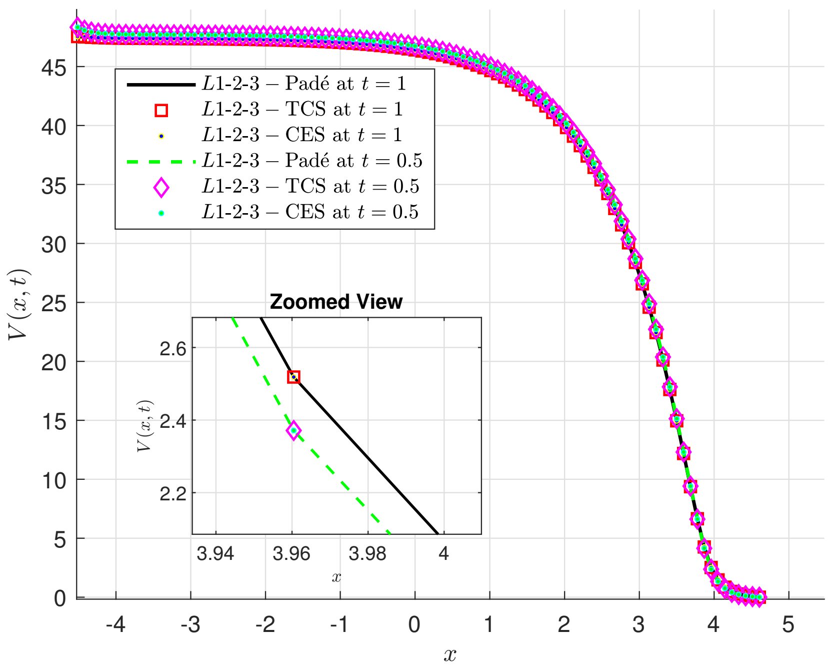


FIGURE 5. European put option price for Example 5.3 at $t = 0.5$ and 1.

6. Conclusion

In this work, we derived three numerical methods to solve the time-fractional Black-Scholes model, with fourth-order accuracy in space and $4-\alpha$ order in time, where $0 < \alpha < 1$. The spatial discretization is conducted using the fourth-order Padé approximation, a fourth-order Taylor compact scheme and a fourth-order compact exponential scheme, combined with $L1$-2-3 discretization for approximating the time-fractional derivative. The solvability of all three schemes is presented. In addition, through rigorous analysis, the convergence and stability of $L1$-2-3-CES and $L1$-2-3-TCS schemes are proved. Finally, numerical experiments are presented verifying the theoretical order of convergence in both space and time, and the accuracy of the proposed schemes is demonstrated.

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