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INCORPORATING THE INFLUENCE OF HUNTING COOPERATION AND COMPETITION INTO THE PREY-PREDATOR MODEL

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Abstract. This mathematical model consists of three nonlinear ordinary differential equations between two prey (N_1, N_2) and only one predator (P). The two prey (N_1, N_2) grow logistically under predator pressure, and the predator depends mainly on the first prey (N_1). According to function response Holling's type II formula, indirect competition exists between N_1 and N_2 because the predator exerts strong pressure on first prey (N_1), while second prey (N_2) is affected more simply. Equilibrium points for the mathematical model between predator and two prey, which stabilize over time, were calculated, and the local stability around the equilibrium points of this proposed model was analyzed using the Lyapunov function. Finally, numerical simulation was used to demonstrate the results.

Keywords: two-prey one-predator model; Holling type II functional response ;stability analysis.

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1. Introduction

Mathematics integrates these models as hypotheses. It helps us understand the behavior of scientific systems, study population interactions that affect each other's growth rates, and develop the skills needed to prepare and manipulate mathematical models. The research presents more than 100 real-world examples [3, 6, 14]. From these mathematical examples, students learn how to formulate, analyze, and estimate models [2, 11, 12]. This section is based on ordinary differential equations accompanied by examples from biology, environment, medicine, and other sciences. In wildlife ecology, the interface between theory and experimentation is still weak, so we specifically look to define a two-prey, one-predator model, where the interaction between species is analyzed in two or three dimensions, and the prey will grow exponentially in the absence of the predator, assuming no threat to the prey other than the multiple predator [1, 5, 10].

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Mathematical models are essential tools for studying ecological relationships among organisms, helping us understand how populations change over time under the influence of reproduction, competition, and predation in an ecosystem. The predator-prey relationship is a dynamic one, in which predators influence the numbers and diversity of their prey [9, 13]. Among these models, the model that describes an ecosystem consisting of a single predator feeding on two prey, with the prey evolving to resist predation, stands out. This relationship achieves balance in the ecosystem by controlling animal populations, maintaining the natural growth of the prey, and mitigating predation pressure from the predator [7, 4, 8, 15].

This paper provides an illustrative example of the nature of the mathematical model. We assume the first prey (rabbits), the second prey (mice), and the predator (foxes). The first and second preys grow logistically through reproduction, but the number of rabbits decreases with increasing density (internal competition), while the number of mice is affected by the number of foxes. As the predator's number increases, its number decreases (below saturation), while the fox consumes part of the rabbit via a function Holling's type II (Predation decreases when full) the number of foxes increases as long as there is sufficient food from rabbits and decreases if prey decreases or natural death increases.

2. Model formulation

In this research, a mathematical model consisting of three-dimensional differential equations is formulated. This model is designed to consist of a first prey (N_1) and a second prey (N_2) representing the population size at that time, and a single predator (P) representing the population size at that time. This model depicts the relationship between the predator and the two prey, a function that takes into account predator saturation when the prey population increases, based on the function Holling's type II. That is, the predator's ability to attack its prey does not increase without limits, but rather reaches its maximum due to the time required to process the prey. Accordingly, the dynamics of this model can be interpreted as shown.

$$\begin{aligned}
\frac{dN_1}{dt} &= N_1\left(r_1\left(1-\frac{N_1}{k_1}\right)-\frac{P(\beta+\gamma P)}{1+a(\beta+\gamma P)N_1}-\alpha_1 N_2\right),\\
\frac{dN_2}{dt} &= N_2\left(r_2\left(1-\frac{N_2}{k_2}\right)-\alpha_2 N_1\right),\\
\frac{dP}{dt} &= P\left(\frac{eP(\beta+\gamma P)}{1+a(\beta+\gamma P)N_1}-d_2\right).
\end{aligned} \tag{1}$$

The variables and parameters of the above system (1) can be described in the following table:

Theorem 2.1. *All the solutions of system (1) which initiate in $\mathbb{R}^3_+$ are uniformly bounded.*

Proof. Let $(N_1(t), N_2(t), P(t))$ be any solution of the system (1) with a non-negative initial condition $(N_1(0), N_2(0), P(0))$. Now consider a function: $M(t) = N_1(t) + N_2(t) + P(t)$. We get:

$$\frac{dM}{dt} < r_1N_1\left(1-\frac{N_1}{k_1}\right)+r_2N_2\left(1-\frac{N_2}{k_2}\right)-dP-N_1N_2\left(\alpha_1+\alpha_2\right)-(1-e)\frac{P(\beta+\gamma P)}{1+a(\beta+\gamma P)N_1}.$$

According to the biological facts, $0 < e < 1$, we get:

TABLE 1. System parameters shown in (1)

Parameters	**Description**
$r_i > 0,\ i = 1, 2$	The logistic growth rate of first and second prey, respectively.
$k_i > 0,\ i = 1, 2$	The carrying capacity of the first and second prey, respectively.
$\alpha_i > 0,\ i = 1, 2$	Predation rate of the predator on both the first and second prey, respectively.
$\beta > 0$	Baseline attack rate, which represents the basic ability of a predator to catch prey.
$a > 0$	Represents predator saturation rate and the extent to which prey handing effects predator saturation.
$d_1 > 0$	Natural mortality rate of a predator.
$\gamma > 0$	Represents the effect of predator density on catch rate.
$e > 0$	The conversion rates of food to a predator.

$$\frac{dM}{dt} < 2r_1N_1 - \frac{r_1N_1^2}{k_1} + 2r_2N_2 - \frac{r_2N_2^2}{k_2} - (dP + r_1N_1 + r_2N_2).$$

When we solve this differential inequality using the comparison theorem, we get:

$$\frac{dM}{dt} \le 2r_1N_1 - \frac{r_1N_1^2}{k_1} + 2r_2N_2 - \frac{r_2N_2^2}{k_2} - (d + \alpha_1 + \alpha_2)\, M.$$

$$\frac{dM}{dt} \le \max(N_1(t)) + \max(N_2(t)) - hM,$$

$$\frac{dM}{dt} + hM \le 2k_1 + 2k_2, \quad \text{where} \quad h = \min\{d, \alpha_1, \alpha_2\}.$$

$$\frac{dM}{dt} + hM \le L, \quad \text{where} \quad L = (2k_1 + 2k_2).$$

Using the comparison theorem to solve this differential equation for the initial value $M(0) = M_0$, we obtain:

$$M(t) \le \frac{L}{h} + \left(M_0 - \frac{L}{h}\right) e^{-ht},$$

Then

$$\lim_{t\to\infty} M(t) \le \frac{L}{h}.$$

Thus, $0 \le M(t) \le \frac{L}{h}, \forall t > 0$. Hence the proof is completed. □

3. Existence of equilibrium points

This section discusses system (1) and defines stability at each possible equilibrium point, which can satisfy the conditions of its existence, as explained below.

(1) The trivial equilibrium point $U_0 = (0, 0, 0)$, always exists.
(2) The first axial equilibrium point $U_1 = (k_1, 0, 0)$ always exists.
(3) The second axial equilibrium point $U_2 = (0, k_2, 0)$ always exists.
(4) The equilibrium point $U_3 = (\widetilde{N}_1, \widetilde{N}_2, 0)$ exists by solving the following set of equations:

$$\begin{aligned} & r_1 - \frac{r_1 N_1}{k_1} - \alpha_1 N_2 \\ & r_2 - \frac{r_2 N_2}{k_2} - \alpha_2 N_1 \end{aligned} \tag{2}$$

From the first equation ofequation of (2) we have,

$$N_2 = \frac{r_1 \left(k_1 - N_1\right)}{\alpha_1 k_1}$$

From the second equation (2) we have,

$$N_1 = \frac{r_2 \left(k_2 - N_2\right)}{\alpha_2 k_2}$$

So, $U_3 = \left(\widetilde{N}_1, \widetilde{N}_2, 0\right)$ exists if the next conditions hold:

$$k_1 > N_1, \quad k_2 > N_2. \tag{3}$$

(1)
(5) The equilibrium point $U_4 = (\overline{N}_1, 0, \overline{P})$ exists by solving the following set of equations:

$$r_1 \left(1 - \frac{N_1}{k_1}\right) - \frac{P(\beta + \gamma P)}{1 + a(\beta + \gamma P) N_1} \tag{4}$$

$$\frac{eP(\beta + \gamma P)}{1 + a\left(\beta + \gamma p\right) N_1} - d = 0, \tag{5}$$

From equation (5) we have:

$$P = \frac{d}{N_1 \gamma (e - ad)} - \frac{\beta}{\gamma}, \tag{6}$$

Now by substituting (6) in (4) we get:

$$f\left(N_1\right) = B_1 N_1^4 + B_2 N_1^3 + B_3 N_1^2 + B_4 N_1 + B_5 = 0,$$

Where:

$$\begin{aligned} B_1 &= \gamma^2 a r_1 \left(e - ad\right)^2 (1 - \beta), \\ B_2 &= \gamma^2 r_1 \left(e - ad\right) \left[\left(e - ad\right) \left[\left(\beta + 1\right) a k_1 - 1\right] - ad\right], \\ B_3 &= \left(e - ad\right) \left[r_1 k_1 \gamma^2 e - \beta^2 \gamma \left(e - ad\right) \left(k_1 - 1\right)\right], \\ B_4 &= \left(e - ad\right) \gamma d \beta k_1 \left(2\gamma - 1\right), \\ B_5 &= -d^2 \gamma k_1 < 0. \end{aligned}$$

So, $U_4 = \left(\bar{N}_1, 0, \bar{P}\right)$ exists if the following conditions hold:

$$\begin{gathered} e > ad, \\ \frac{d}{N_1 \gamma \left(e - ad\right)} > \frac{\beta}{\gamma}, \\ 1 < \min\left\{\left(\beta + 1\right) a k_1, 2\gamma, k_1\right\}, \\ \left(e - ad\right) \left[\left(\beta + 1\right) a k_1 - 1\right] > ad, \\ r_1 k_1 \gamma^2 e > \beta^2 \gamma \left(e - ad\right) \left(k_1 - 1\right). \end{gathered}$$

Finally, the positive equilibrium point $U_5 = \left(\dot{N}_1, \dot{N}_2, \dot{P}\right)$ exists by solving the following set of equations:

$$r_1\left(1-\frac{N_1}{k_1}\right)-\frac{P\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma P\right)N_1}-\alpha_1 N_2=0, \tag{7}$$

$$r_2\left(1-\frac{N_2}{k_2}\right)-\alpha_2 N_1=0, \tag{8}$$

$$\frac{eP(\beta+\gamma P)}{1+a\left(\beta+\gamma P\right)N_1}-d_2=0, \tag{9}$$

From equation (7) we have:

$$N_2=\frac{r_1\left(k_1-N_1\right)\left(1+a\left(\beta+\gamma P\right)N_1\right)-k_1 P\left(\beta+\gamma P\right)}{\alpha_1 k_1\left(1+a\left(\beta+\gamma P\right)N_1\right)}, \tag{10}$$

From equation (9) we have:

$$N_1=\frac{d}{\left(e-ad\right)\left(\beta+\gamma P\right)}, \tag{11}$$

Now by substituting (11) and (10) in (8) we get:

$$f\left(P\right)=V_1 P^3+V_2 P^2+V_3 P+V_4=0,$$

Where:

$$V_1=k_1 k_2 \gamma^2\left(e\left(2ad\beta-e\right)+a^2 d^2 \beta\right),$$

$$\begin{aligned} V_2 =&\gamma k_1 k_2\left[\beta^2\left(ad\left(3e-2ad\right)-2e^2\right)+\alpha_1\gamma\left(ad^2\left(3e-2ad\right)+\right.\right.\\ &\left.\left.2ad\left(e-ad\right)-e^2 d\right)\right]+\gamma^2\left[r_1 r_2 k_1\left(e^2\left(e-ad\right)\right)+ead\beta^2\right], \end{aligned}$$

$$V_3=r_1 r_2 \gamma\left[k_1\beta\left(ad\left(3e-2ad\right)-2e^2\right)-ed\right]+\beta^3 k_1 k_2\left[ad\left(e-ad\right)-e^2\right],$$

$$\begin{aligned} V_4 =&k_1 k_1 \beta\left[r_2\beta\left(ad\left(ad+e\right)+a^2 d^2\gamma+\alpha_1\left(e^2+ad\right)d+d^2\alpha_1\alpha_2\left(e-2ad\right)\right)\right]\\ &+r_1 r_2 \beta\left[k_1\left(ad\left(1+ad\gamma\right)-de+k_1\left(e^2+a^2 d^2\right)\right)\right], \end{aligned}$$

So, $U_5 = \left(\dot{N}_1, \dot{N}_2, \dot{P}\right)$ exists if, in addition to condition (3), the following conditions hold:

$$r_1\left(k_1-N_1\right)\left(1+a\left(\beta+\gamma P\right)N_1\right)>k_1 P\left(\beta+\gamma P\right),$$

$$e>\text{Max}\left\{ad, 2ad\beta, \frac{2ad}{3}, \frac{a^2 d^2 \beta}{\left(2ad\beta-e\right)}\right\},$$

$$ad>\text{Max}\left\{\frac{2e^2}{\left(3e-2ad\right)}, \frac{e^2 d}{2\left(e-ad\right)}, \frac{ed}{\left(1+ad\right)}\right\},$$

$$k_1\beta\left(ad\left(3e-2ad\right)-2e^2\right)>ed,$$

4. Local stability analysis

This section discusses the local stability of system (1) through the calculation of the Jacobian matrix $J(N_1, N_2, P)$ of the system (1). as follows: $J = [n_{ij}]_{3\times 3}$,

Where:

$$
\begin{aligned}
n_{11} &= r_1 - \frac{2r_1N_1}{k_1} - \alpha_1 N_2 - \frac{P(\beta+\gamma P)}{(1+a(\beta+\gamma P)N_1)^2},\\
n_{12} &= -\alpha_1 N_1,\\
n_{13} &= \frac{-\left[\beta + aN_1\left(\beta^2+\gamma^2P^2\right) + 2P\gamma(1+\beta N_1)\right]}{(1+a(\beta+\gamma P)N_1)^2},\\
n_{21} &= -\alpha_2 N_2,\\
n_{22} &= r_2 - \frac{2r_2N_2}{k_2} - \alpha_2 N_1,\\
n_{23} &= 0,\\
n_{31} &= \frac{eP(\beta+\gamma P)}{(1+a(\beta+\gamma P)N_1)^2},\\
n_{32} &= 0,\\
n_{33} &= \frac{e\left[\beta + aN_1\left(\beta^2+\gamma^2P^2\right) + 2P\gamma(1+\beta N_1)\right]}{(1+a(\beta+\gamma P)N_1)^2} - d.
\end{aligned}
$$

4.1. **Local stability of U_0.** The Jacobian matrix at $U_0 = (0,0,0)$ is given by:

$$
J_0 = J(U_0) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & d \end{bmatrix},
$$

The characteristic equation of J_0 is then given by:

$$
(r_1-\lambda)(r_2-\lambda)(d-\lambda) = 0,
$$

which gives:

$$
\lambda_1 = r_1 > 0, \quad \lambda_2 = r_2 > 0, \quad \lambda_3 = d > 0,
$$

Hence U_0 is unstable.

4.2. **Local stability of U_1.** The Jacobian matrix at $U_1 = (k_1, 0, 0)$ is given by:

$$
J_1 = \begin{bmatrix} -r_1 & -\alpha_1 k_1 & \frac{k_1\left[\beta + ak_1\beta^2\right]}{(1+a\beta k_1)^2} \\ 0 & r_2 & 0 \\ 0 & 0 & \frac{k_1 e\left[\beta + ak_1\beta^2\right]}{(1+a\beta k_1)^2} - d \end{bmatrix},
$$

The eigenvalues of J_1 are:

$$
(-r_1-\lambda_1)(r_2-\lambda_1)\left(\frac{k_1 e\left[\beta + ak_1\beta^2\right]}{(1+a\beta k_1)^2} - d - \lambda_1\right) = 0
$$

which gives

$$
\lambda_{1N_1} = -r_1 < 0, \quad \lambda_{1N_2} = r_2 > 0, \quad \lambda_{1P} = \frac{k_1 e\left[\beta + ak_1\beta^2\right]}{(1+a\beta k_1)^2} - d,
$$

Hence U_1 is a saddle point (unstable).

4.3. **Local stability of U_2.** The Jacobian matrix at

$$U_2 = (0, k_2, 0)$$

is given by:

$$J_2 = J(U_2) = \begin{bmatrix} r_1 - \alpha_1 k_2 & 0 & 0 \\ -\alpha_2 k_2 & -r_2 & 0 \\ 0 & 0 & -d \end{bmatrix},$$

Then the eigenvalues of J_2 are:

$$(r_1 - \alpha_1 k_2 - \lambda_2)(-r_2 - \lambda_0)(-d - \lambda_0) = 0,$$

which gives: $\lambda_{2N_1} = r_1 - \alpha_1 k_2$, $\lambda_{2N_2} = -r_2 < 0$, $\lambda_{2P} = -d < 0$

Therefore, U_2 will be locally asymptotically stable if the following condition is met:

$$r_1 < \alpha_1 k_2.$$

4.4. **Local stability of U_3.** The Jacobian matrix of $U_3 = (\widetilde{N}_1, \widetilde{N}_2, 0)$ is given by

$$J_3 = J(U_3) = [v_{ij}]_{3\times 3},$$

Where

$$\begin{aligned}
v_{11} &= r_1 - \frac{2r_1\widetilde{N}_1}{k_1} - \alpha_1\widetilde{N}_2, \\
v_{12} &= -\alpha_1\widetilde{N}_1, \\
v_{13} &= \frac{\widetilde{N}_1\left[\beta + a\widetilde{N}_2\beta^2\right]}{\left(1 + a\beta\widetilde{N}_1\right)^2}, \\
v_{21} &= -\alpha_2\widetilde{N}_2, \\
v_{22} &= r_2 - \frac{2r_2\widetilde{N}_2}{k_2} - \alpha_2\widetilde{N}_1, \\
v_{23} &= 0, \\
v_{31} &= 0, \\
v_{32} &= 0, \\
v_{33} &= \frac{\widetilde{N}_1 e\left[\beta + a\widetilde{N}_2\beta^2\right]}{\left(1 + a\beta\widetilde{N}_1\right)^2} - d.
\end{aligned}$$

The characteristic equation of J_3 is then given by:

$$\lambda^2 + \lambda\left(r_1 - \frac{2r_1\widetilde{N}_1}{k_1} - \alpha_1\widetilde{N}_2 + r_2 - \frac{2r_2\widetilde{N}_2}{k_2} - \alpha_2\widetilde{N}_1\right) +$$

$$\left(r_1 - \frac{2r_1\widetilde{N}_1}{k_1} - \alpha_1\widetilde{N}_2\right)\left(r_2 - \frac{2r_2\widetilde{N}_2}{k_2} - \alpha_2\widetilde{N}_1\right) - \alpha_1\alpha_2\widetilde{N}_2\widetilde{N}_1$$

$=0$

So, either

$$\frac{\widetilde{N}_1 e\left[\beta + a\widetilde{N}_2\beta^2\right]}{\left(1+a\beta\widetilde{N}_1\right)^2} - d - \lambda_{3P} = 0,$$

which gives:

$$\lambda_{3P} = \frac{\widetilde{N}_1 e\left[\beta + a\widetilde{N}_2\beta^2\right]}{\left(1+a\beta\widetilde{N}_1\right)^2} - d,$$

or

$$\lambda^2 + \lambda\left(r_1\left(1-\frac{2\widetilde{N}_1}{k_1}\right) - \alpha_1\widetilde{N}_2 + r_2\left(1-\frac{2\widetilde{N}_2}{k_2}\right) - \alpha_2\widetilde{N}_1\right) +$$

$$\left(r_1 - \frac{2r_1\widetilde{N}_1}{k_1} - \alpha_1\widetilde{N}_2\right)\left(r_2 - \frac{2r_2\widetilde{N}_2}{k_2} - \alpha_2\widetilde{N}_1\right) - \alpha_1\alpha_2\widetilde{N}_2\widetilde{N}_1 = 0,$$

which gives:

$$\lambda_{3N_1} + \lambda_{3N_2} = r_1\left(1-\frac{2\widetilde{N}_1}{k_1}\right) - \alpha_1\widetilde{N}_2 + r_2\left(1-\frac{2\widetilde{N}_2}{k_2}\right) - \alpha_2\widetilde{N}_1.$$

$$\lambda_{3N_1}\cdot\lambda_{3N_2} = \left(r_1\left(1-\frac{2\widetilde{N}_1}{k_1}\right) - \alpha_1\widetilde{N}_2\right)\left(r_2\left(1-\frac{2\widetilde{N}_2}{k_2}\right) - \alpha_2\widetilde{N}_1\right) - \alpha_1\alpha_2\widetilde{N}_2\widetilde{N}_1.$$

Therefore, U_3 will be locally asymptotically stable if the following conditions are met.

$$\frac{\widetilde{N}_1 e\left[\beta + a\widetilde{N}_2\beta^2\right]}{\left(1+a\beta\widetilde{N}_1\right)^2} > d,$$

$$1 < \frac{2\widetilde{N}_1}{k_1},$$

$$1 < \frac{2\widetilde{N}_2}{k_2},$$

$$\left(r_1\left(1-\frac{2\widetilde{N}_1}{k_1}\right) - \alpha_1\widetilde{N}_2\right)\left(r_2\left(1-\frac{2\widetilde{N}_2}{k_2}\right) - \alpha_2\widetilde{N}_1\right) > \alpha_1\alpha_2\widetilde{N}_2\widetilde{N}_1. \tag{12}$$

Otherwise it is a saddle point.

4.5. **Local stability of U_4.** The Jacobian matrix at $U_4 = (\bar{N}_1, 0, \bar{P})$ is given by:

$$J_4 = J(U_4) = [s_{ij}]_{3\times 3},$$

Where:

$$\delta_{11} = r_1\left(1 - \frac{2\bar{N}_1}{k_1}\right) - \frac{\bar{P}(\beta+\gamma\bar{P})}{\left(1+a\left(\beta+\gamma P\right)N_1\right)^2},$$
$$\delta_{13} = \frac{-\left[\beta + aN_1\left(\beta^2+\gamma^2P^2\right) + 2P\gamma\left(1+\beta N_1\right)\right]}{\left(1+a\left(\beta+\gamma P\right)N_1\right)^2},$$
$$\delta_{12} = -\alpha_1\bar{N}_1,$$
$$\delta_{21} = 0,$$
$$\delta_{22} = r_2 - \alpha_2\bar{N}_2,$$
$$\delta_{23} = 0,$$
$$\delta_{31} = \frac{e\bar{P}(\beta+\gamma\bar{P})}{\left(1+a\left(\beta+\gamma P\right)N_1\right)^2},$$
$$\delta_{33} = \frac{e\left[\beta + aN_1\left(\beta^2+\gamma^2P^2\right) + 2P\gamma\left(1+\beta N_1\right)\right]}{\left(1+a\left(\beta+\gamma P\right)N_1\right)^2} - d.$$

The characteristic equation of J_4 is then given by:

$$\left[\lambda^3 + \bar{B}_1\lambda^2 + \bar{B}_2\lambda + \bar{B}_3\right] = 0, \tag{12}$$

$$\bar{B}_1 = -\left(s_{11}+s_{22}+s_{33}\right) > 0,$$
$$\bar{B}_2 = \left(s_{22}\right)\left(s_{11}+s_{33}\right) - \left(s_{13}\right)\left(s_{31}\right) + (s_{11})(s_{33}) > 0,$$
$$\bar{B}_3 = \left(s_{22}\right)\left[\left(s_{13}\right)\left(s_{31}\right) + \left(s_{11}\right)\left(s_{33}\right)\right] > 0,$$

According to the Routh-Hurwitz criterion, equation (12) has roots with negative real parts, if $\bar{B}_i > 0,\ i = 1,2,3$ and $\Delta = B_1B_2 - B_3 > 0$. Clearly, $\bar{B}_i > 0$ if the following conditions hold:

$$1 < \frac{2\bar{N}_1}{k_1},$$
$$r_2 < \alpha_2\bar{N}_2,$$
$$\frac{e\left[\beta + a\bar{N}_1\left(\beta^2+\gamma^2\bar{P}^2\right) + 2\bar{P}\gamma\left(1+\beta\bar{N}_1\right)\right]}{\left(1+a\left(\beta+\gamma\bar{P}\right)\bar{N}_1\right)^2} < d,$$

Straightforward computation shows that: $\Delta = (\bar{Q}_1\bar{B}_1 - \bar{Q}_2)$, where:

$$\bar{Q}_1 = -\Big[s_{33}^2\left(s_{22}+s_{11}\right) + 3\left(s_{11}\right)\left(s_{22}\right)\left(s_{33}\right) + s_{22}^2\left(s_{11}+s_{33}\right) + s_{11}^2\left(s_{22}+s_{33}\right) - s_{31}s_{13}\left(s_{11}+s_{22}+s_{33}\right)\Big]$$

and

$$\bar{Q}_2 = \left(s_{22}^2\right)\left[s_{31}^2s_{13}^2 + s_{11}^2s_{33}^2 - \left(s_{11}\right)\left(s_{33}\right)\left(s_{31}\right)(s_{13})\right],$$

Hence, Δ will be positive if in addition to conditions [2.m-2.o] the following condition holds:

$$\bar{Q}_1\bar{B}_1 > \bar{Q}_2,$$

So, U_4 is locally stable. It's unstable otherwise.

4.6. **Local stability of U_5.** The Jacobian matrix at $U_5 = (\dot{N}_1, \dot{N}_2, \dot{P})$ is given by:

$$J_5 = J(U_5) = [y_{ij}]_{3\times 3},$$

$$y_{11} = r_1 - \frac{2r_1\dot{N}_1}{k_1} - \alpha_1\dot{N}_2 - \frac{\dot{P}\left(\beta+\gamma\dot{P}\right)}{\left(1+a\left(\beta+\gamma\dot{P}\right)\dot{N}_1\right)^2},$$
$$y_{12} = -\alpha_1\dot{N}_1,$$
$$y_{13} = \frac{-\left[\beta + a\dot{N}_1\left(\beta^2+\gamma^2\dot{P}^2\right) + 2\dot{P}\gamma\left(1+\beta\dot{N}_1\right)\right]}{\left(1+a\left(\beta+\gamma\dot{P}\right)\dot{N}_1\right)^2},$$
$$y_{21} = -\alpha_2\dot{N}_2,$$
$$y_{22} = r_2 - \frac{2r_2\dot{N}_2}{k_2} - \alpha_2\dot{N}_1,$$
$$y_{23} = 0,$$
$$y_{31} = \frac{e\dot{P}\left(\beta+\gamma\dot{P}\right)}{\left(1+a\left(\beta+\gamma\dot{P}\right)\dot{N}_1\right)^2},$$
$$y_{32} = 0,$$
$$y_{33} = \frac{e\left[\beta + a\dot{N}_1\left(\beta^2+\gamma^2\dot{P}^2\right) + 2\dot{P}\gamma\left(1+\beta\dot{P}_1\right)\right]}{\left(1+a\left(\beta+\gamma\dot{P}\right)\dot{N}_1\right)^2} - d,$$

The characteristic equation of J_5 is then given by:

$$\left[\lambda^3 + \dot{C}_1\lambda^2 + \dot{C}_2\lambda + \dot{C}_3\right] = 0, \tag{13}$$

$$\dot{C}_1 = -\left(y_{11}+y_{22}+y_{33}\right) > 0,$$
$$\dot{C}_2 = (y_{11})(y_{22}+y_{33}) + (y_{22})(y_{33}) - (y_{13})(y_{31}) - (y_{21})(y_{12}) > 0,$$
$$\dot{C}_3 = (y_{22})(y_{13})(y_{31}) + (y_{21})(y_{12})(y_{33}) > 0,$$

According to the Routh-Hurwitz criterion, equation (13) has roots with negative real parts, if $\dot{C}_i > 0,\ i = 1,2,3$ and $\Delta = C_1C_2 - C_3 > 0$. Clearly, $\dot{C}_i > 0$ if the following conditions hold:

$$1 < \frac{2\dot{N}_1}{k_1},$$
$$1 < \frac{2\dot{N}_2}{k_2},$$
$$\frac{e\left[\beta + a\dot{N}_1\left(\beta^2+\gamma^2\dot{P}^2\right) + 2\dot{P}\gamma\left(1+\beta\dot{N}_1\right)\right]}{\left(1+a\left(\beta+\gamma\dot{P}\right)\dot{N}_1\right)^2} < d, \tag{14}$$
$$(y_{22})(y_{13})(y_{31}) > (y_{21})(y_{12})(y_{33}),$$

Straightforward computation shows that: $\dot{\Delta} = (\dot{A}_1\dot{C}_3 - \dot{A}_2)$, where:

$$\dot{A}_1 = -\Big[y_{11}^2 (y_{33} + y_{22}) + 3 (y_{11}) (y_{22}) (y_{33}) + y_{22}^2 (y_{11} + y_{33}) + y_{33}^2 (y_{11} + y_{22}) - (y_{11} + y_{22} + y_{33}) (y_{21}y_{12} + y_{31}y_{13})\Big]$$

$$\dot{A}_2 = y_{31}^2 y_{13}^2 y_{22}^2 + 2y_{21}y_{12}y_{31}y_{13}y_{22}y_{33} + y_{21}^2 y_{12}^2 y_{33}^2,$$

Hence, Δ will be positive if in addition to conditions [12-14] the following conditions hold:

$$y_{21}y_{12} > y_{31}y_{13},$$
$$y_{31}^2 y_{13}^2 y_{22}^2 + y_{21}^2 y_{12}^2 y_{33}^2 > 2y_{21}y_{12}y_{31}y_{13}y_{22}y_{33},$$
$$\dot{A}_1\dot{C}_3 > \dot{A}_2,$$

So, U_5 is locally stable. It's unstable otherwise.

5. Global stability analysis

This section examines the global stability of the equilibrium points of the system (1) using the Lyapunov function as shown below.

Theorem 5.1. *The point U_2 is a globally stable equilibrium point approximately within the range in $\mathbb{R}_+^3$ and that satisfies the next conditions:*

$$\frac{r_1k_1}{4} < \frac{r_2}{k_2}\left(N_2 - \bar{\bar{N}}_2\right)^2 + N_1N_2 (\alpha_1 + \alpha_2) + \alpha_2 N_1\bar{\bar{N}}_2. \quad (15)$$

Proof. Consider the following function:

$$T_1 (N_1, N_2, P) = N_1 + \left(N_2 - \bar{\bar{N}}_2 - \bar{\bar{N}}_2 \ln \frac{N_2}{\bar{\bar{N}}_2}\right) + P.$$

Clearly $T_1 : \mathbb{R}_+^3 \to \mathbb{R}$ is a $T_1 \in C^1$ since the function is a positive determinant, its derivative T_1 with respect to time t, using the equations in the system, gives the following form

$$\frac{dT_1}{dt} = -\left[\frac{r_2}{k_2}\left(N_2 - \bar{\bar{N}}_2\right)^2 + N_1N_2 (\alpha_1 + \alpha_2) + \alpha_2 N_1\bar{\bar{N}}_2\right] - \frac{N_1P (\beta + \gamma P)}{1 + a (\beta + \gamma p) N_1} (1 - e) + r_1N_1 \left(1 - \frac{N_1}{k_1}\right).$$

Now since the function $f (N_1) = r_1N_1 \left(1 - \frac{N_1}{k_1}\right)$ in the second term represents a logistic function with respect to N_1 and hence it is bounded above by the constant $\frac{r_1k_1}{4}$, then according to the biological facts, $(1 > e)$.

Hence,

$$\frac{dT_1}{dt} < -\left[\frac{r_2}{k_2}\left(N_2 - \bar{\bar{N}}_2\right)^2 + N_1N_2 (\alpha_1 + \alpha_2) + \alpha_2 N_1\bar{\bar{N}}_2\right] + \frac{r_1k_1}{4}.$$

So, $\frac{dT_1}{dt} < 0$ according to condition (15). Hence U_2 is globally asymptotically stable. □

Theorem 5.2. *The point U_3 is a globally stable equilibrium point approximately within the range in $\mathbb{R}_+^3$ and that satisfies the next conditions:*

$$N_i > \widetilde{N}_i, \ i = 1, 2 \quad (16)$$

$$\widetilde{R}_1 > \widetilde{R}_2, \quad (17)$$

Where:

$$\widetilde{R}_1 = -\left[\frac{r_1}{k_1}\left(N_1 - \widetilde{N}_1\right)^2 + \left(N_1 - \widetilde{N}_1\right)\left(N_2 - \widetilde{N}_2\right)(\alpha_1 + \alpha_2) + \frac{r_2}{k_2}\left(N_2 - \widetilde{N}_2\right)^2\right],$$

$$\widetilde{R}_2 = -\frac{\widetilde{N}_1 P (\beta + \gamma P)}{1 + a(\beta + \gamma p) N_1}.$$

Proof. Consider the following function:

$$T_2(N_1, N_2, P) = \left(N_1 - \widetilde{N}_1 - \widetilde{N}_1 \ln \frac{N_1}{\widetilde{N}_1}\right) + \left(N_2 - \widetilde{N}_2 - \widetilde{N}_2 \ln \frac{N_2}{\widetilde{N}_2}\right) + P.$$

Clearly $T_2 : \mathbb{R}^3_+ \to \mathbb{R}$ is a $T_2 \in C^1$ since the function is a positive determinant, its derivative T_2 with respect to time t, using the equations in the system, gives the following form

$$\frac{dT_2}{dt} = -\left[\frac{r_1}{k_1}\left(N_1 - \widetilde{N}_1\right)^2 + \left(N_1 - \widetilde{N}_1\right)\left(N_2 - \widetilde{N}_2\right)(\alpha_1 + \alpha_2) + \frac{r_2}{k_2}\left(N_2 - \widetilde{N}_2\right)^2\right] - \frac{N_1 P (\beta + \gamma P)}{1 + a(\beta + \gamma p) N_1}(1 - e) + \frac{\widetilde{N}_1 P (\beta + \gamma P)}{1 + a(\beta + \gamma p) N_1}.$$

So, according to the biological facts in theorem (1), always $(1 > e)$.

$$\frac{dT_2}{dt} < -\widetilde{R}_1 + \widetilde{R}_2.$$

So, $\frac{dT_2}{dt} < 0$ according to conditions (16-17). Hence U_3 is globally asymptotically stable. □

Theorem 5.3. *The point U_4 is a globally stable equilibrium point approximately within the range in $\mathbb{R}^3_+$ and that satisfies the next conditions:*

$$\bar{R}_1 > \bar{R}_2, \tag{18}$$

Where:

$$\bar{R}_1 = -\left[\frac{r_1}{k_1}\left(N_1 - \bar{N}_1\right)^2 + \frac{e N_1 \bar{P}(\beta + \gamma P)}{1 + a(\beta + \gamma P) N_1} + \frac{e \bar{N}_1 P (\beta + \gamma P)}{1 + a(\beta + \gamma P)\bar{N}_1} + (\alpha_1 + \alpha_2) N_1 N_2 + \alpha_1 N_2 \bar{N}_1\right],$$

$$\bar{R}_2 = \frac{N_1 \bar{P}(\beta + \gamma P)}{1 + a(\beta + \gamma P)\bar{N}_1} + \frac{\bar{N}_1 P (\beta + \gamma P)}{1 + a(\beta + \gamma P) N_1} + \frac{r_2 k_2}{4}. \tag{1}$$

Proof. Consider the following function:

$$T_3(N_1, N_2, P) = \left(N_1 - \bar{N}_1 - \bar{N}_1 \ln \frac{N_1}{\bar{N}_1}\right) + N_2 + \left(P - \bar{P} - \bar{P} \ln \frac{P}{\bar{P}}\right).$$

Clearly $T_3 : \mathbb{R}^3_+ \to \mathbb{R}$ is a $T_3 \in C^1$ since the function is a positive determinant, its derivative T_3 with respect to time t, using the equations in the system, gives the following form:

$$\begin{aligned}\frac{dT_3}{dt} = -\Bigg[&\frac{r_1}{k_1}\left(N_1 - \bar{N}_1\right)^2 + \frac{eN_1\bar{P}\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)N_1} + \frac{e\bar{N}_1P\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)\bar{N}_1}\\ &+\alpha_1N_2\bar{N}_1 + \left(\alpha_1+\alpha_2\right)N_1N_2\Bigg] + \frac{N_1\bar{P}\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)\bar{N}_1}\\ &+\frac{\bar{N}_1P\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)N_1} - (1-e)\frac{\bar{N}_1\bar{P}\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)\bar{N}_1}\\ &-(1-e)\frac{N_1P\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)N_1} + r_2N_2\left(1-\frac{N_2}{k_2}\right).\end{aligned}$$

Now since the function $f\left(N_2\right) = r_2N_2\left(1-\frac{N_2}{k_2}\right)$ in the second term represents a logistic function with respect to N_2 and hence it is bounded above by the constant $\frac{r_2k_2}{4}$, then according to the biological facts, $(1 > e)$.

Hence,

$$\begin{aligned}\frac{dT_3}{dt} < -\Bigg[&\frac{r_1}{k_1}\left(N_1 - \bar{N}_1\right)^2 + \frac{eN_1\bar{P}\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)N_1} + \frac{e\bar{N}_1P\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)\bar{N}_1}\\ &+\alpha_1N_2\bar{N}_1 + \left(\alpha_1+\alpha_2\right)N_1N_2\Bigg] + \frac{N_1\bar{P}\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)\bar{N}_1}\\ &+\frac{\bar{N}_1P\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)N_1} - (1-e)\frac{\bar{N}_1\bar{P}\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)\bar{N}_1}\\ &-(1-e)\frac{N_1P\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma p\right)N_1} + \frac{r_2k_2}{4}.\end{aligned}$$

Then $\frac{dT_3}{dt} = -\bar{R}_1 + \bar{R}_2$. So, $\frac{dT_3}{dt} < 0$ according to condition (18). Hence U_4 is globally asymptotically stable. □

Theorem 5.4. *The point U_5 is a globally stable equilibrium point approximately within the range in $\mathbb{R}^3_+$ and that satisfies the next conditions:*

$$N_i > \dot{N}_i,\ i = 1, 2 \tag{19}$$

$$\dot{R}_1 > \dot{R}_2, \tag{20}$$

Where:

$$\begin{aligned}\dot{R}_1 = -\Bigg[&\frac{r_1}{k_1}\left(N_1-\dot{N}_1\right)^2 + \left(N_1-\dot{N}_1\right)\left(N_2-\dot{N}_2\right)\left(\alpha_1+\alpha_2\right)\\ &+\frac{r_2}{k_2}\left(N_2-\dot{N}_2\right)^2 + \frac{e\dot{N}_1\dot{P}(\beta+\gamma P)}{1+a(\beta+\gamma P)\dot{N}_1} + \frac{eN_1\dot{P}(\beta+\gamma P)}{1+a(\beta+\gamma P)N_1}\Bigg]\end{aligned}$$

$$\dot{R}_2 = \frac{\dot{N}_1P\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma P\right)\dot{N}_1} + \frac{N_1\dot{P}\left(\beta+\gamma P\right)}{1+a\left(\beta+\gamma P\right)N_1}$$

Proof. Consider the following function:

$$T_4\left(N_1, N_2, P\right)=\left(N_1-\dot{N}_1-\dot{N}_1 \ln \frac{N_1}{\dot{N}_1}\right)+\left(N_2-\dot{N}_2-\dot{N}_2 \ln \frac{N_2}{\dot{N}_2}\right)+\left(P-\dot{P}-\dot{P} \ln \frac{P}{\dot{P}}\right).$$

Clearly $T_4 : \mathbb{R}_{+}^{3} \rightarrow \mathbb{R}$ is a $T_4 \in C^1$ since the function is a positive determinant, its derivative T_4 with respect to time t, using the equations in the system, gives the following form:

$$\begin{aligned}
\frac{dT_4}{dt} = &-\left[\frac{r_1}{k_1}\left(N_1-\dot{N}_1\right)^2+\left(N_1-\dot{N}_1\right)\left(N_2-\dot{N}_2\right)\left(\alpha_1+\alpha_2\right)\right. \\
&\left.+\frac{r_2}{k_2}\left(N_2-\dot{N}_2\right)^2\right]-(1-e)\left(\frac{\dot{N}_1 \dot{P}(\beta+\gamma P)}{1+a(\beta+\gamma p) \dot{N}_1}+\frac{N_1 P(\beta+\gamma P)}{1+a(\beta+\gamma p) N_1}\right) \\
&+\frac{\dot{N}_1 P(\beta+\gamma P)}{1+a(\beta+\gamma p) \dot{N}_1}+\frac{N_1 \dot{P}(\beta+\gamma P)}{1+a(\beta+\gamma p) \dot{N}_1} \\
&-\frac{e \dot{N}_1 \dot{P}(\beta+\gamma P)}{1+a(\beta+\gamma p) \dot{N}_1}-\frac{e N_1 \dot{P}(\beta+\gamma P)}{1+a(\beta+\gamma p) N_1}.
\end{aligned}$$

So, according to the biological facts in theorem (1), always $(1>e)$.

$$\begin{aligned}
\frac{dT_4}{dt} < &-\left[\frac{r_1}{k_1}\left(N_1-\dot{N}_1\right)^2+\left(N_1-\dot{N}_1\right)\left(N_2-\dot{N}_2\right)\left(\alpha_1+\alpha_2\right)+\frac{r_2}{k_2}\left(N_2-\dot{N}_2\right)^2\right. \\
&\left.+\frac{e \dot{N}_1 \dot{P}(\beta+\gamma P)}{1+a(\beta+\gamma p) \dot{N}_1}+\frac{e N_1 \dot{P}(\beta+\gamma P)}{1+a(\beta+\gamma p) N_1}\right] \\
&+\frac{\dot{N}_1 P(\beta+\gamma P)}{1+a(\beta+\gamma p) \dot{N}_1}+\frac{N_1 \dot{P}(\beta+\gamma P)}{1+a(\beta+\gamma p) \dot{N}_1}.
\end{aligned}$$

Then

$$\frac{dT_4}{dt}=-\dot{R}_1+\dot{R}_2.$$

So, $\frac{dT_4}{dt}<0$ according to conditions (19-20). Hence U_5 is globally asymptotically stable. □

6. Numerical simulations

In this section, a numerical simulation of the overall dynamic behavior of the proposed model (1) is presented using MATLAB to verify the validity of the analytical results. Numerical simulation can be performed using a different set of tools. This provides us with a comprehensive explanation of the effect of changing the values of the system parameters.

$$\begin{aligned}
&r_1=0.3,\ k_1=0.6,\ \beta=0.1,\ \gamma=0.025,\ a=0.5,\ \alpha_1=0.7, \\
&r_2=0.3,\ k_2=0.6,\ \alpha_2=0.7,\ e=0.4,\ d_2=0.01.
\end{aligned}$$

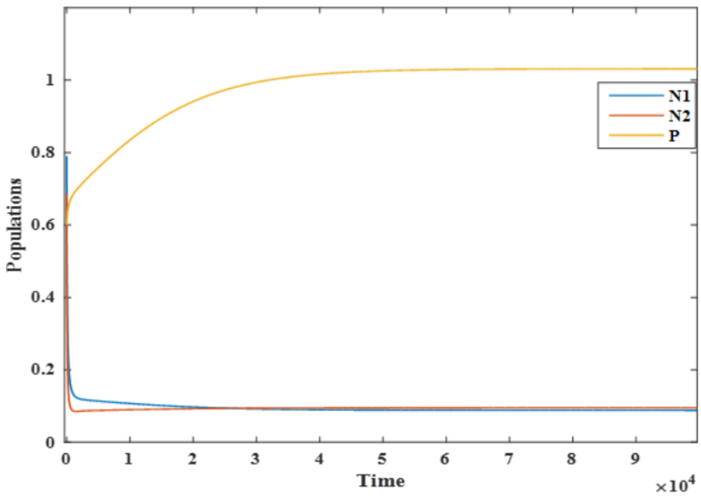

FIGURE 1. Graphical representation of the dynamics of the three types according to the data provided in Equation (1.7), where the system approaches a positive equilibrium point

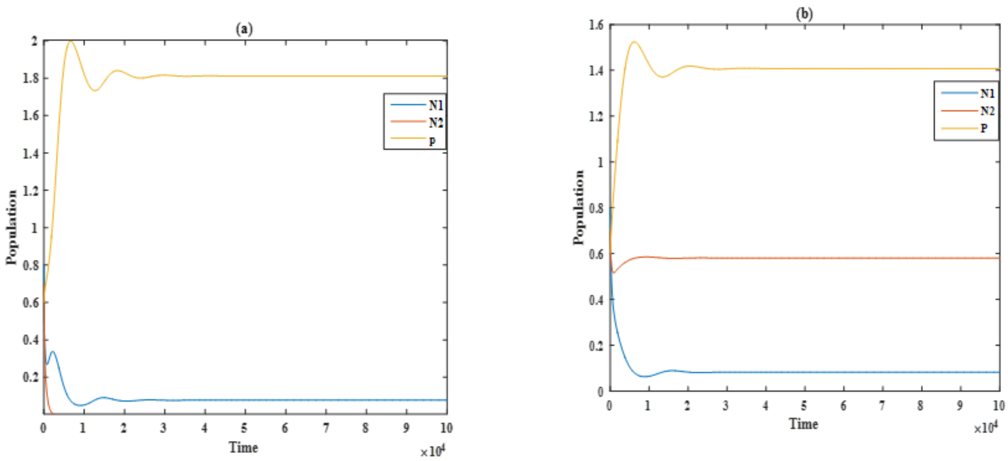

FIGURE 2. Graphical representation of the coefficient values mentioned in Equation (1.6) for the coefficient k_2. As the values of the coefficient change in the range $k_2 > 0.01$, the system approaches the equilibrium point U_4, as shown in Fig. (a). As the range increases $0.12 < k_2 < 0.25$, the solution approaches the equilibrium point approximately U_5, as shown in Fig. (b).

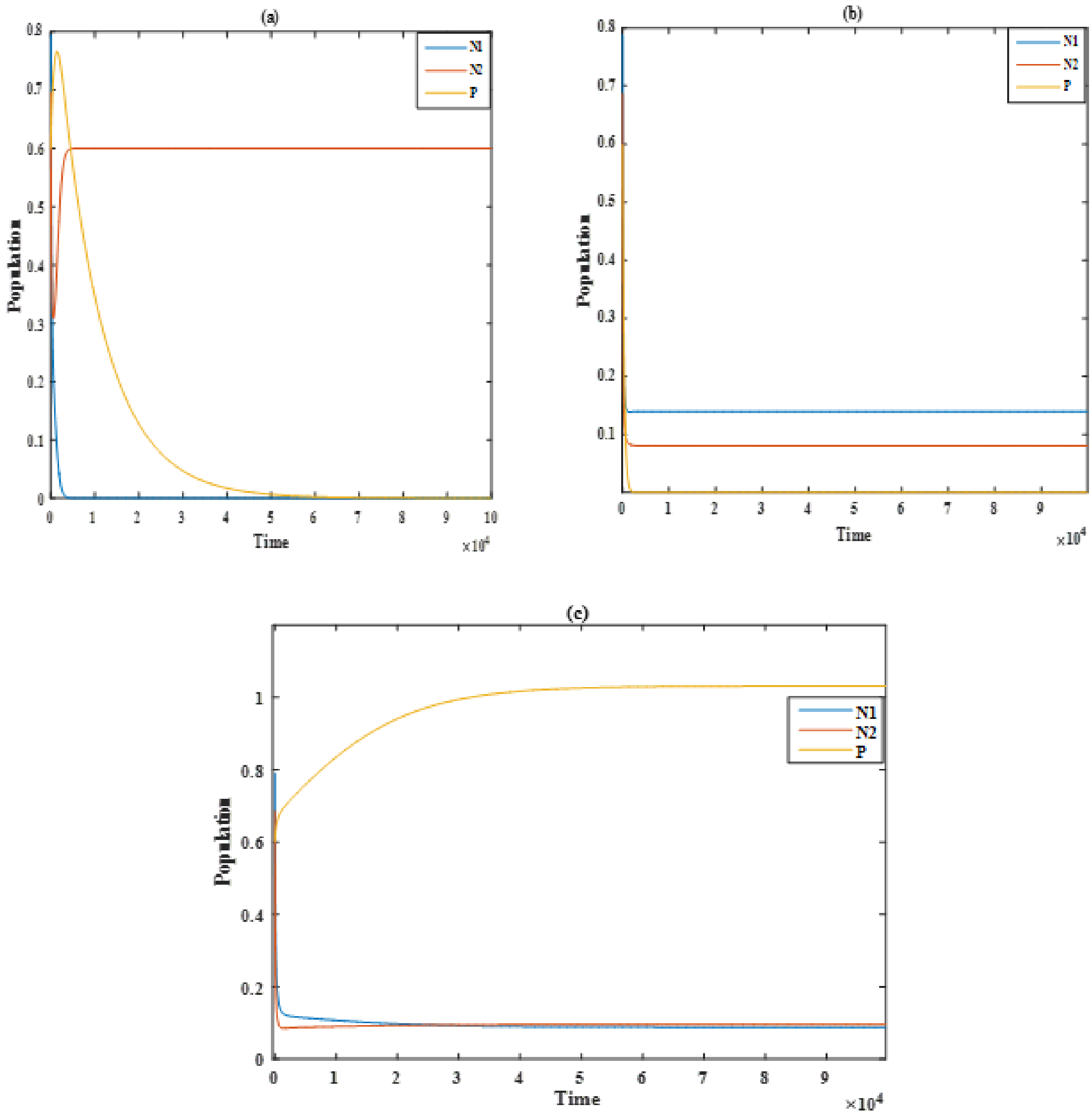

FIGURE 3. Graphical representation of the coefficient values mentioned in Equation (1.6) for the coefficient k_2. We observe that the system approaches the equilibrium point U_2 when the values of the coefficient change in the range $0.07 < \alpha_2 < 0.18$, as shown in Fig. (a). As the solution approaches the equilibrium point U_3 when the range increases to $0.18 \leq \alpha_2 < 0.25$, as shown in Fig. (b). However, for $0.25 \leq \alpha_2 \leq 2$, the solution approaches the positive equilibrium point U_5, as shown in Fig. (c).

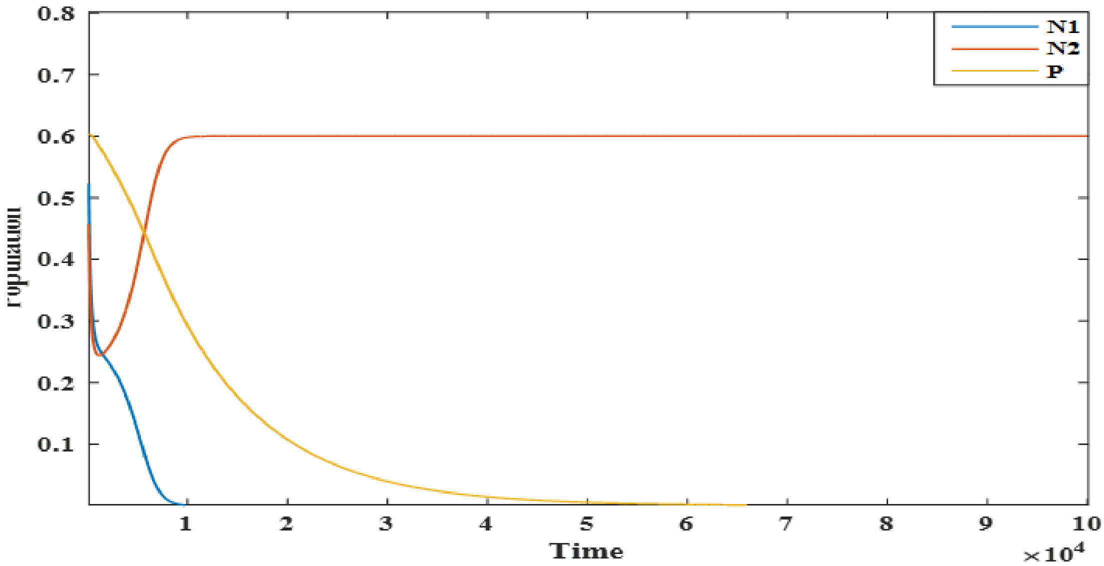

FIGURE 4. Graphical representation of the dynamics shows that the system is approaching the equilibrium point U_2, which represents the death of the first prey and the predator.

7. Conclusions and discussions

This study is based on a mathematical model consisting of three nonlinear ordinary differential equations that describe the relationship between the two prey (N_1, N_2) and only one predator (P). The two prey (N_1, N_2) grow logistically under predator pressure, and the predator depends mainly on the first prey (N_1). According to Holling's type II functional response formula, indirect competition exists between (N_1) and (N_2) because the predator exerts strong pressure on first prey (N_1), while second prey (N_2) is affected more simply.

Equilibrium points for the mathematical model between predator and two prey, which stabilize over time, were calculated, and the local stability around the equilibrium points of this proposed model was analyzed using the Lyapunov function. Finally, it was solved numerically using the set of default parameter values given.

Therefore, for system (1), we obtain the following results, which can be summarized as follows: There is no periodic dynamics for system (1); some parameters play an important role in the dynamics of the system (1), while for other parameters the solution approaches a positive equilibrium point.

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