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$(\psi, \mathcal{GF})$-CONTRACTION MAPPINGS AND RELATED FIXED POINT RESULTS IN PARTIAL MODULAR METRIC SPACES

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Abstract. In this paper, we focus on establishing the existence of fixed points results for $(\psi, \mathcal{GF})$-contraction mapping in partial modular metric spaces. In support of this result, a suitable example is given. For the application, we demonstrate how these results can be utilized to investigate the existence and uniqueness of solutions for a system of $\mathcal{V}$olterra-type integral equation. Moreover, we demonstrate the existence of solutions of fractional differential equations in the framework of partial modular metric spaces.

Keywords: Partial modular metric space, (ψ, F)-contraction mappings, $\mathcal{V}$olterra integral equations, fractional differential equations.

AMS Subject Classification: 47H10; 54H25.

1. Introduction

$\mathcal{M}$atthews [14] proposed the idea of partial metric space, which generalizes the metric space by allowing a non-zero self-distance. $\mathcal{C}$histyakov [6] introduced modular metric space (MMS). Researchers enrich fixed point theory in these spaces (see [3, 4, 7, 11, 12],[15]-[19]). There are numerous applications for partial metric space (PMS) and modular metric space (MMS) in both pure and applied mathematics viz., the solution's existence and uniqueness for integral equations, for Riemann-Liouville fractional differential equations, for problems in dynamical programming and so on. New researchers are drawn to this field because of its remarkable applicability in a variety of fields.

$\mathcal{H}$osseinzadeh and $\mathcal{P}$arvaneh [10] proposed the idea of partial modular metric spaces (PMMSs) that generalize PMS, as well as established several fixed point theorems in such new spaces. Recently, Das et al. [8] refined the idea of PMMS to remove the discrepancy in non-zero self distances and triangular inequality in PMMS [10]. Also, include some examples and a few fixed point results with applications. PMMS serves as a very powerful and useful tool in the study of non-linear processes.

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Wardowski [22] introduced F-contraction to generalize Banach fixed point theorem. Later in [23], Wardowski enlarges this F-contraction as (ψ, F)-contraction in metric spaces. Rossafi et al. [13] introduced the concept of $(\psi, \mathcal{MF})$-contraction in C^*-algebra valued metric spaces.

Inspired by Wardowski in [23] and $\mathcal{R}$ossafi et al. in [13], this paper introduces a new contraction mapping $(\psi, \mathcal{GF})$ in PMMS. A few common fixed point results utilizing $(\psi, \mathcal{GF})$-contraction are demonstrated in this space. Also, provided examples to validate the results. The solution's existence for a given system of $\mathcal{V}$olterra integral equations and fractional differential equations is discussed as an application.

The paper is divided into four sections. The first and second sections cover the introduction of PMMS, $(\psi, \mathcal{GF})$-contraction mapping, and existing fixed point results. The third section contains some new generalized fixed point results and examples. The fourth section discusses $\mathcal{V}$olterra integral equations and fractional differential equations applications. The final section contains the paper's conclusion.

2. Preliminaries:

In the present section, we review a few definitions and characteristics which are used in our results. Hosseinzadeh and Parvaneh [10] introduced PMMS as follows:

Definition 2.1. [10] *Let $\mathcal{N} \neq \emptyset$. A function $\omega^p : (0, +\infty) \times \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ specified by $\omega^p(\lambda, \varrho_1, \varrho_2) = \omega^p_\lambda(\varrho_1, \varrho_2)$, is said to be partial modular metric (PMM) on $\mathcal{N}$ if it meets the conditions stated below:*

(ω^p_1): $\omega^p_\lambda(\varrho_1, \varrho_2) = \omega^p_\lambda(\varrho_1, \varrho_1) = \omega^p_\lambda(\varrho_2, \varrho_2) \Leftrightarrow \varrho_1 = \varrho_2$, $\forall \lambda > 0$;
(ω^p_2): $\omega^p_\lambda(\varrho_1, \varrho_1) \leq \omega^p_\lambda(\varrho_1, \varrho_2)$, $\forall \varrho_1, \varrho_2 \in \mathcal{N}$ *and* $\forall \lambda > 0$;
(ω^p_3): $\omega^p_\lambda(\varrho_1, \varrho_2) = \omega^p_\lambda(\varrho_2, \varrho_1)$, $\forall \varrho_1, \varrho_2 \in \mathcal{N}$ *and* $\forall \lambda > 0$;
(ω^p_4): $\omega^p_{\lambda+\mu}(\varrho_1, \varrho_2) \leq \omega^p_\lambda(\varrho_1, \varrho_3) + \omega^p_\mu(\varrho_3, \varrho_2) - \left[\frac{\omega^p_\lambda(\varrho_1,\varrho_1)+\omega^p_\lambda(\varrho_3,\varrho_3)+\omega^p_\mu(\varrho_3,\varrho_3)+\omega^p_\mu(\varrho_2,\varrho_2)}{2}\right]$, $\forall \varrho_1, \varrho_2 \in \mathcal{N}$ *and* $\forall \lambda, \mu > 0$.

In recent times, Das et al. [8] refined the definition of PMMS [10], to remove the discrepancy in the non-zero self-distance and triangularity conditions, as follows:

Definition 2.2. [8] *Let $\mathcal{N} \neq \emptyset$. A function $\omega^p : (0, +\infty) \times \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ defined by $\omega^p(\lambda, \varrho_1, \varrho_2) = \omega^p_\lambda(\varrho_1, \varrho_2)$, is said to be PMM on $\mathcal{N}$ if it meets the following conditions:*

$(\omega^p_{1/})$: $\omega^p_\lambda(\varrho_1, \varrho_1) = \omega^p_\mu(\varrho_1, \varrho_1)$ *and* $\omega^p_\lambda(\varrho_1, \varrho_1) = \omega^p_\lambda(\varrho_1, \varrho_2) = \omega^p_\lambda(\varrho_2, \varrho_2) \Leftrightarrow \varrho_1 = \varrho_2$, $\forall \lambda, \mu > 0$;

$(\omega^p_{2/})$: $\omega^p_\lambda(\varrho_1, \varrho_1) \leq \omega^p_\lambda(\varrho_1, \varrho_2)$, $\forall \varrho_1, \varrho_2 \in \mathcal{N}$ *and* $\forall \lambda > 0$;

$(\omega^p_{3/})$: $\omega^p_\lambda(\varrho_1, \varrho_2) = \omega^p_\lambda(\varrho_2, \varrho_1)$, $\forall \varrho_1, \varrho_2 \in \mathcal{N}$ *and* $\forall \lambda > 0$;

$(\omega^P_{4/})$: $\omega^p_{\lambda+\mu}(\varrho_1, \varrho_2) \leq \omega^p_\lambda(\varrho_1, \varrho_3) + \omega^p_\mu(\varrho_3, \varrho_2) - \omega^p_\lambda(\varrho_3, \varrho_3)$, $\forall \varrho_1, \varrho_2 \in \mathcal{N}$ *and* $\lambda, \mu > 0$.

Obviously, if $\omega^p_\lambda(\varrho_1, \varrho_2) = 0$, from $(\omega^p_{1/})$ and $(\omega^p_{2/})$, we get $\varrho_1 = \varrho_2$, but the converse might not hold. If PMM ω^p on $\mathcal{N}$ has a finite value and it does not depend on the parameter $\lambda > 0$ which is $\omega^p_\lambda(\varrho_1, \varrho_2) = \omega^p_\mu(\varrho_1, \varrho_2)$, $\forall \lambda, \mu > 0$, then $p(\varrho_1, \varrho_2) = \omega^p_\lambda(\varrho_1, \varrho_2)$ is a partial metric (PM) on $\mathcal{N}$.

Definition 2.3. [8] *If along with the conditions $(\omega^p_{1/})$, $(\omega^p_{2/})$ and $(\omega^p_{3/})$, a PMM ω^p on $\mathcal{N}$ satisfies the following:*

$(\omega^p_{5/})$: $\omega^p_{\lambda+\mu}(\varrho_1, \varrho_2) \leq \frac{\lambda}{\lambda+\mu}\omega^p_{\lambda}(\varrho_1, \varrho_2) + \frac{\mu}{\lambda+\mu}\omega^p_{\mu}(\varrho_3, \varrho_2) - \frac{\lambda}{\lambda+\mu}\omega^p_{\lambda}(\varrho_3, \varrho_3)$,

$\forall \varrho_1, \varrho_2, \varrho_3 \in \mathcal{N}$ *and* $\forall \lambda, \mu > 0$.

Then it is called convex.

Definition 2.4. [8] *For a given* $\varrho_0 \in \mathcal{N}$,

$$\mathcal{N}_{\omega^p}(\varrho_0) = \{\delta \in \mathcal{N} : \lim_{\lambda \to +\infty} \omega^p_{\lambda}(\varrho_0, \delta) = \kappa\},$$

for some $\kappa \geq 0$ *and*

$$\mathcal{N}^*_{\omega^p}(\varrho_0) = \{\delta \in \mathcal{N} : \exists \lambda = \lambda(\delta) > 0, \, \omega^p_{\lambda}(\varrho_0, \delta) < +\infty\}.$$

Then two sets $\mathcal{N}_{\omega^p}$ *and* $\mathcal{N}^*_{\omega^p}$ *are said to be partial modular spaces (PMS) centred at* ϱ_0*. Its clear that* $\mathcal{N}_{\omega^p} \subset \mathcal{N}^*_{\omega^p}$*. Then,* $\mathcal{N}_{\omega^p} \equiv \mathcal{N}_{\omega^p}(\varrho_0)$ *and* $\mathcal{N}^* \equiv \mathcal{N}^*_{\omega^p}(\varrho_0)$*, if* $\varrho_0 \in \mathcal{N}$ *is an arbitrary.*

Example 2.1. [8] *Let* $\mathcal{N} = \mathbb{R}$*. We define*
$\omega^p_{\lambda}(\varrho_1, \varrho_2) = e^{-\lambda} \mid \varrho_1 - \varrho_2 \mid + \mid \varrho_1 \mid + \mid \varrho_2 \mid$, $\forall \varrho_1, \varrho_2 \in \mathcal{N}$ *and* $\forall \lambda > 0$.

Then ω^p *is a PMM on* $\mathcal{N}$*.*

Definition 2.5. [8] *If* ω^p *denote a PMM on* $\mathcal{N} \neq \emptyset$ *and* $\{\varrho_{\Bbbk}\}$ *be a sequence in PMMS* $\mathcal{N}_{\omega^p}$*, then*

(i) $\{\varrho_{\Bbbk}\}$ *converges to* $\delta \in \mathcal{N}_{\omega^p}$*, if and onle if for every* $\epsilon > 0$, $\exists \, \Bbbk_0 \in \mathbb{N} \cup \{0\}$ *such that*
$$\mid \omega^p_{\lambda}(\varrho_{\Bbbk}, \delta) - \omega^p_{\lambda}(\delta, \delta) \mid \leq \epsilon,$$
$\forall \quad \Bbbk \geq \Bbbk_0$ *and* $\forall \quad \lambda > 0$*. We then write* $\lim_{\Bbbk \to +\infty} \omega^p_{\lambda}(\varrho_{\Bbbk}, \delta) = \omega^p_{\lambda}(\delta, \delta)$, $\forall \lambda > 0$;

(ii) $\{\varrho_{\Bbbk}\}$ *is said to be Cauchy in* $\mathcal{N}_{\omega^p}$ *if* $\lim_{\Bbbk, m \to +\infty} \omega^p_{\lambda}(\varrho_{\Bbbk}, \varrho_m) = \mathfrak{c}$, $\forall \lambda > 0$*, for some* $\mathfrak{c} \geq 0$*. Then,* $\lim_{\Bbbk \to +\infty} \omega^p_{\lambda}(\varrho_{\Bbbk}, \varrho_{\Bbbk}) = \lim_{m \to +\infty} \omega^p_{\lambda}(\varrho_m, \varrho_m) = \mathfrak{c}$*. Hence,* $\mathfrak{c} = 0$ *if* $\{\varrho_{\Bbbk}\}$ *is a Cauchy sequence in* $\mathcal{N}_{\omega^s}$*;*

(iii) $\mathcal{N}_{\omega^p}$ *is called complete if every Cauchy sequence converges to some* $\delta \in \mathcal{N}_{\omega^p}$ *such that*

$$\lim_{\Bbbk, m \to +\infty} \omega^p_{\lambda}(\varrho_{\Bbbk}, \varrho_m) = \omega^p_{\lambda}(\delta, \delta), \, \forall \lambda > 0.$$

Definition 2.6. [6] *Let* $\mathcal{N} \neq \emptyset$*. A function* $\omega^s : (0, +\infty) \times \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ *defined by* $\omega^s(\lambda, \varrho_1, \varrho_2) = \omega^s_{\lambda}(\varrho_1, \varrho_2)$*, is said to be MM on* $\mathcal{N}$ *if it meets the following conditions:*

(ω^s_1): $\omega^s_{\lambda}(\varrho_1, \varrho_2) = 0 \Leftrightarrow \varrho_1 = \varrho_2$, $\forall \lambda > 0$;

(ω^s_2): $\omega^s_{\lambda}(\varrho_1, \varrho_2) = \omega^s_{\lambda}(\varrho_2, \varrho_1)$, $\forall \varrho_1, \varrho_2 \in \mathcal{N}$ *and* $\forall \lambda > 0$*;*

(ω^s_3): $\omega^s_{\lambda+\mu}(\varrho_1, \varrho_2) \leq \omega^s_{\lambda}(\varrho_1, \varrho_3) + \omega^s_{\mu}(\varrho_3, \varrho_2)$, $\forall \varrho_1, \varrho_2 \in \mathcal{N}$ *and* $\lambda, \mu > 0$.

Moreover, For a given $\varrho_0 \in \mathcal{N}$,

$$\mathcal{N}_{\omega^s}(\varrho_0) = \{\delta \in \mathcal{N} : \lim_{\lambda \to +\infty} \omega^s_{\lambda}(\varrho_0, \delta) = 0,$$

for some $\kappa \geq 0$*. Then the set* $\mathcal{N}_{\omega^s}$ *is said to be modular spaces centred at* ϱ_0*.*

Lemma 2.1. [8] *Let* $\mathcal{N}_{\omega^p}$ *be a partial modular metric space. Define*

$$\omega^s_{\lambda}(\varrho_1, \varrho_2) = 2\omega^p_{\lambda}(\varrho_1, \varrho_2) - \omega^p_{\lambda}(\varrho_1, \varrho_1) - \omega^p_{\lambda}(\varrho_2, \varrho_2), \; \lambda > 0.$$

Then $\mathcal{N}_{\omega^s}$ *is a modular metric space.*

Lemma 2.2. [8] *Let $\mathcal{N}_{\omega^p}$ be a partial modular metric space. Then*

(i) *$\{x_n\}$ is a partial modular Cauchy sequence in $\mathcal{N}_{\omega^p}$ if and only if it is Cauchy in the modular metric space $\mathcal{N}_{\omega^s}$;*

(ii) *A partial modular metric space $\mathcal{N}_{\omega^p}$ is complete if and only if modular metric space $\mathcal{N}_{\omega^s}$ is complete.*

Definition 2.7. [23] *A mapping $T : \mathcal{N} \to \mathcal{N}$ is referred to as (ψ, F)-contraction if there exists functions $F : (0, +\infty) \to \mathbb{R}$ and $\phi : (0, +\infty) \to (0, +\infty)$ that satisfies the conditions*

(F1) *$\alpha_1 > \alpha_2 \Rightarrow F(\alpha_1) > F(\alpha_2)$, for all $\alpha_1, \alpha_2 > 0$.*

(F2) *for any sequence $\{\alpha_n\}_{n=1}^{+\infty} \subset (0, +\infty)$, where $\alpha_n \to 0$, if and only if $F(\alpha_n) \to -\infty$.*

(F3) *$\liminf_{s\to\alpha^+} \phi(\alpha) > 0$ for all $\alpha > 0$.*

(F4) *$F(d(T\varrho_1, T\varrho_2)) + \phi(d(\varrho_1, \varrho_2)) \leq F(d(\varrho_1, \varrho_2))\ \forall \varrho_1, \varrho_2 \in \mathcal{N}$; such that $T\varrho_1 \neq T\varrho_2$.*

3. Main Results

Inspired by Wardowski [22] and [23], we construct a $(\psi, \mathcal{GF})$-contraction mapping as follows:

Definition 3.1. *Let $\mathcal{S}, \mathcal{T} : \mathcal{N}_{\omega^p} \to \mathcal{N}_{\omega^p}$ be two self-mappings, and is referred to as $(\psi, \mathcal{GF})$-contraction if there exists functions $\mathcal{F} : [0, +\infty) \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ that satisfies the conditions:*

($\mathcal{F}_1$) *$\mathcal{F}$ is continuous and nondecreasing, such that $\alpha_1 > \alpha_2 \Rightarrow \mathcal{F}(\alpha_1) > \mathcal{F}(\alpha_2)$, for all $\alpha_1, \alpha_2 > 0$. Moreover, $\mathcal{F}(\alpha_1) = 0$ if and only if $\alpha_1 = 0$.*

($\mathcal{F}_2$) *ψ is continuous, increasing and positive i.e., $\psi(\alpha_1) > 0$ for all $\alpha_1 \in (0, \infty)$ with $\psi(0) = 0$.*

($\mathcal{F}_3$) *for all $\delta_1, \delta_2 \in \mathcal{N}_{\omega^p}$, and $\lambda > 0$*

$$\omega^p_\lambda(\mathcal{T}\delta_1, \mathcal{S}\delta_2) \geq 0 \Rightarrow \mathcal{F}(\omega^p_\lambda(\mathcal{T}\delta_1, \mathcal{S}\delta_2)) + \psi(\mathcal{G}(\delta_1, \delta_2)) \leq \mathcal{F}(\mathcal{G}(\delta_1, \delta_2)),$$

where, $\mathcal{G}(\delta_1, \delta_2) = \max\{\omega^p_\lambda(\delta_1, \delta_2), \omega^p_\lambda(\delta_1, \mathcal{T}\delta_1), \omega^p_\lambda(\delta_2, \mathcal{S}\delta_2), \frac{[\omega^p_{2\lambda}(\delta_1, \mathcal{S}\delta_2) + \omega^p_{2\lambda}(\delta_2, \mathcal{T}\delta_1)]}{2}\}$.

Theorem 3.1. *Let $(\mathcal{N}_{\omega^p}, \omega^p_\lambda)$ denote a complete PMMS and $\mathcal{S}, \mathcal{T} : \mathcal{N}_{\omega^p} \to \mathcal{N}_{\omega^p}$ be the two self-mappings which satisfies $(\psi, \mathcal{GF})$-contraction mapping with $\mathcal{T}\mathcal{N}_{\omega^p} \subseteq \mathcal{S}\mathcal{N}_{\omega^p}$. Then $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point.*

Proof. Let $\delta_0 \in \mathcal{N}_{\omega^p}$ be a point. Let $\{\mathcal{S}\delta_{\jmath+1}\}$ be a sequence specified by $\mathcal{T}\delta_{2\jmath} = \delta_{2\jmath+1}$ and $\mathcal{S}\delta_{2\jmath+1} = \delta_{2\jmath+2}$, for every $n = 0, 1, 2, \ldots$ If $\delta_m = \delta_{m+1}$ for all $m = 0, 1, 2, \ldots$, then for $m = 2\jmath$, we have

$$\begin{aligned}
\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath+1}) &= \max\{\omega^p_\lambda(\delta_{2\jmath}, \delta_{2\jmath+1}), \omega^p_\lambda(\delta_{2\jmath}, \mathcal{T}\delta_{2\jmath}), \omega^p_\lambda(\delta_{2\jmath+1}, \mathcal{S}\delta_{2\jmath+1}),\\
&\qquad \frac{[\omega^p_{2\lambda}(\delta_{2\jmath}, \mathcal{S}\delta_{2\jmath+1}) + \omega^p_{2\lambda}(\delta_{2\jmath+1}, \mathcal{T}\delta_{2\jmath})]}{2}\}\\
&= \max\{\omega^p_\lambda(\delta_{2\jmath}, \delta_{2\jmath}), \omega^p_\lambda(\delta_{2\jmath}, \delta_{2\jmath}), \omega^p_\lambda(\delta_{2\jmath}, \delta_{2\jmath+2}),\\
&\qquad \frac{[\omega^p_{2\lambda}(\delta_{2\jmath}, \delta_{2\jmath+2}) + \omega^p_{2\lambda}(\delta_{2\jmath}, \delta_{2\jmath})]}{2}\}\\
&= \omega^p_\lambda(\delta_{2\jmath}, \delta_{2\jmath+2}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{F}(\omega^p_\lambda(\mathcal{T}\delta_{2\jmath}, \mathcal{S}\delta_{2\jmath+1})) \leq \mathcal{F}(\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath+1})) - \psi(\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath+1}))\\
&\Rightarrow \mathcal{F}(\omega^p_\lambda(\delta_{2\jmath}, \delta_{2\jmath+2})) \leq \mathcal{F}(\omega^p_\lambda(\delta_{2\jmath}, \delta_{2\jmath+2})) - \psi(\omega^p_\lambda(\delta_{2\jmath}, \delta_{2\jmath+2})).
\end{aligned}$$

As ψ is positive, so, a contradiction arises. Therefore, $\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+2}) = 0$. We get, $\delta_{2\jmath} = \delta_{2\jmath+1} = \delta_{2\jmath+2}$. Similarly, we have $\delta_{2\jmath} = \delta_{2\jmath+1} = \delta_{2\jmath+2} = \delta_{2\jmath+3} = \ldots$, and so on. $\delta_{2\jmath} = \mathcal{T}\delta_{2\jmath} = \mathcal{S}\delta_{2\jmath}$, i.e., $\delta_{2\jmath}$ is a common fixed point.
Suppose, $\delta_m \neq \delta_{m+1}, \forall m = 0, 1, 2, \ldots$ Assume, $\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath-1}) > 0$ we have

$$\begin{aligned}\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath-1}) &= \max\{\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath-1}), \omega_\lambda^p(\delta_{2\jmath}, \mathcal{T}\delta_{2\jmath}), \omega_\lambda^p(\delta_{2\jmath-1}, \mathcal{S}\delta_{2\jmath-1}),\\ &\quad \frac{[\omega_{2\lambda}^p(\delta_{2\jmath}, \mathcal{S}\delta_{2\jmath-1}) + \omega_{2\lambda}^p(\delta_{2\jmath-1}, \mathcal{T}\delta_{2\jmath})]}{2}\}\\ &= \max\{\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath-1}), \omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1}), \omega_\lambda^p(\delta_{2\jmath-1}, \delta_{2\jmath}),\\ &\quad \frac{[\omega_{2\lambda}^p(\delta_{2\jmath}, \delta_{2\jmath}) + \omega_{2\lambda}^p(\delta_{2\jmath-1}, \delta_{2\jmath+1})]}{2}\}\\ &= \max\{\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath-1}), \omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1}), \omega_\lambda^p(\delta_{2\jmath-1}, \delta_{2\jmath}),\\ &\quad \frac{[\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath}) + \omega_\lambda^p(\delta_{2\jmath-1}, \delta_{2\jmath}) + \omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1}) - \omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath})]}{2}\}\\ &= \max\{\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath-1}), \omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1})\}.\end{aligned}$$

Let $\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath-1}) = \omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1})$, then

$$\begin{aligned}&\mathcal{F}(\omega_\lambda^p(\mathcal{T}\delta_{2\jmath}, \mathcal{S}\delta_{2\jmath-1})) \leq \mathcal{F}(\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath-1})) - \psi(\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath-1}))\\ \Rightarrow\ &\mathcal{F}(\omega_\lambda^p(\delta_{2\jmath+1}, \delta_{2\jmath})) \leq \mathcal{F}(\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1})) - \psi(\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1})).\end{aligned}$$

As ψ is positive, hence a contradiction arise.
So, $\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath-1}) = \omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath-1})$, and we get

$$\begin{aligned}&\mathcal{F}(\omega_\lambda^p(\mathcal{T}\delta_{2\jmath}, \mathcal{S}\delta_{2\jmath-1})) \leq \mathcal{F}(\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath-1})) - \psi(\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath-1}))\\ \Rightarrow\ &\mathcal{F}(\omega_\lambda^p(\delta_{2\jmath+1}, \delta_{2\jmath})) \leq \mathcal{F}(\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath-1})) - \psi(\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath-1}))\\ &\qquad < \mathcal{F}(\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath-1})). \end{aligned} \tag{1}$$

Similarly, we can also show that,

$$\begin{aligned}&\mathcal{F}(\omega_\lambda^p(\mathcal{T}\delta_{2\jmath}, \mathcal{S}\delta_{2\jmath+1})) \leq \mathcal{F}(\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath+1})) - \psi(\mathcal{G}(\delta_{2\jmath}, \delta_{2\jmath+1}))\\ \Rightarrow\ &\mathcal{F}(\omega_\lambda^p(\delta_{2\jmath+1}, \delta_{2\jmath+2})) \leq \mathcal{F}(\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1})) - \psi(\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1}))\\ &\qquad < \mathcal{F}(\omega_\lambda^p(\delta_{2\jmath}, \delta_{2\jmath+1})). \end{aligned} \tag{2}$$

As, $\mathcal{F}$ is non decreasing so, from (1) and (2) we have, $\omega_\lambda^p(\delta_m, \delta_{m+1})$ is monotonically decreasing sequence to non-negative real numbers.
Let, $\exists\, 0 \leq \beta$, such that $\lim_{n\to+\infty} \omega_\lambda^p(\delta_m, \delta_{m+1}) = \beta$, $\forall \lambda > 0$. Hence, from (1) and (2), we have $\lim_{n\to+\infty} \mathcal{F}(\omega_\lambda^p(\delta_m, \delta_{m+1})) = 0$, $\forall \lambda > 0$, i.e.,

$$\lim_{n\to+\infty} \omega_\lambda^p(\delta_m, \delta_{m+1}) = 0, \forall \lambda > 0. \tag{3}$$

We will now demonstrate that $\{\delta_m\}$ is a cauchy sequence in $\mathcal{N}_{\omega^p}$. It is enough to show that $\{\mathcal{S}\delta_{m+1}\}$ is a cauchy sequence in $\mathcal{N}_{\omega^s}$.
Now, $0 \leq \omega_\lambda^p(\delta_m, \delta_m) \leq \omega_\lambda^p(\delta_m, \delta_{m+1})$, $\forall \lambda > 0$, we have

$$\lim_{\jmath\to+\infty} \omega_\lambda^p(\delta_m, \delta_m) = 0. \tag{4}$$

Also, $0 \leq \omega_\lambda^p(\delta_{m+1}, \delta_{m+1}) \leq \omega_\lambda^p(\delta_{m+1}, \delta_{m+2})$ implies

$$\lim_{\jmath\to+\infty} \omega_\lambda^p(\delta_{m+1}, \delta_{m+1}) = 0, \quad \forall \lambda > 0. \tag{5}$$

If possible, let $\{\delta_m\}$ be not a Cauchy in $\mathcal{N}_{\omega^s}$. Then there is $\epsilon > 0$ and subsequences $\{\delta_{m(\kappa)}\}$ and $\{\delta_{\jmath(\kappa)}\}$ along with $\jmath(\kappa) > m(\kappa) > k$ such that

$$\omega^s_\lambda(\delta_{m(\kappa)}, \delta_{\jmath(\kappa)}) > \epsilon,\ \forall\ \lambda > 0.$$

We choose $\jmath(\kappa)$ for $m(\kappa)$, such that its the least integer along with $\jmath(\kappa) > m(\kappa)$ that satisfies the above inequality. Hence

$$\omega^s_\lambda(\delta_{m(\kappa)}, \delta_{\jmath(\kappa)-1}) \leq \epsilon,\ \forall\ \lambda > 0.$$

We have

$$\begin{aligned}\omega^s_\lambda(\delta_{m(\kappa)}, \delta_{\jmath(\kappa)}) &\leq \omega^s_{\frac{\lambda}{2}}(\delta_{m(\kappa)}, \delta_{\jmath(\kappa)-1}) + \omega^s_{\frac{\lambda}{2}}(\delta_{\jmath(\kappa)-1}, \delta_{\jmath(\kappa)}) \\ &\leq \epsilon + \omega^s_{\frac{\lambda}{2}}(\delta_{\jmath(\kappa)-1}, \delta_{\jmath(\kappa)}). \end{aligned} \tag{6}$$

Again, from Lemma 2.1;

$$\omega^s_\lambda(\delta_{\jmath(\kappa)-1}, \delta_{\jmath(\kappa)}) = 2\omega^p_\lambda(\delta_{\jmath(\kappa)-1}, \delta_{\jmath(\kappa)}) - \omega^p_\lambda(\delta_{\jmath(\kappa)-1}, \delta_{\jmath(\kappa)-1}) - \omega^p_\lambda(\delta_{\jmath(\kappa)}, \delta_{\jmath(\kappa)}). \tag{7}$$

Using (3),(4) and (5), we have

$$\lim_{k\to+\infty} \omega^s_\lambda(\delta_{\jmath(\kappa)-1}, \delta_{\jmath(\kappa)}) = 0. \tag{8}$$

Using (6) and (8), we get

$$\epsilon < \lim_{k\to+\infty} \omega^s_\lambda(\delta_{m(\kappa)}, \delta_{\jmath(\kappa)}) < \epsilon + 0.$$

This implies

$$\lim_{k\to+\infty} \omega^s_\lambda(\delta_{m(\kappa)}, \delta_{\jmath(\kappa)}) = \epsilon. \tag{9}$$

Similarly we prove that,

$$\lim_{k\to+\infty} \omega^s_\lambda(\delta_{m(\kappa)-1}, \delta_{\jmath(\kappa)-1}) = \epsilon.$$

Thus

$$\begin{aligned}&\lim_{k\to+\infty} \omega^p_\lambda(\delta_{m(\kappa)-1}, \delta_{\jmath(\kappa)-1}) \\ &= \frac{1}{2} \lim_{k\to+\infty} [2\omega^p_\lambda(\delta_{m(\kappa)-1}, \delta_{\jmath(\kappa)-1}) - \omega^p_\lambda(\delta_{\jmath(\kappa)-1}, \delta_{\jmath(\kappa)-1}) - \omega^p_\lambda(\delta_{m(\kappa)-1}, \delta_{m(\kappa)-1})] \\ &= \frac{1}{2} \lim_{k\to+\infty} \omega^s_\lambda(\delta_{m(\kappa)-1}, \delta_{\jmath(\kappa)-1}) \\ &= \frac{\epsilon}{2} \\ &= b\,(say).\end{aligned}$$

Similarly, we obtain

$\lim_{k\to+\infty} \omega^p_\lambda(\delta_{m(\kappa)-1}, \delta_{\jmath(\kappa)-1}) = \frac{\epsilon}{2} = b$ and $\lim_{k\to+\infty} \omega^p_\lambda(\delta_{m(\kappa)}, \delta_{\jmath(\kappa)}) = \frac{\epsilon}{2} = b$.

Now from triangular inequality,

$$\begin{aligned}\omega^p_\lambda(\delta_{m(\kappa)}, \delta_{\jmath(\kappa)}) \leq\ &\omega^p_{\frac{\lambda}{k}}(\delta_{m(\kappa)}, \delta_{m(\kappa)+1}) + \omega^p_{\frac{\lambda}{k}}(\delta_{m(\kappa)+1}, \delta_{m(\kappa)+2}) + \ldots + \omega^p_{\frac{\lambda}{k}}(\delta_{\jmath(\kappa)-1}, \delta_{\jmath(\kappa)}) \\ &- [\omega^p_{\frac{\lambda}{k}}(\delta_{m(\kappa)+1}, \delta_{m(\kappa)+1}) + \omega^p_{\frac{\lambda}{k}}(\delta_{m(\kappa)+2}, \delta_{m(\kappa)+2}) + \ldots + \omega^p_{\frac{\lambda}{k}}(\delta_{\jmath(\kappa)-1}, \delta_{\jmath(\kappa)-1})].\end{aligned}$$

Using (3), (4) and (5) we have

$$\lim_{\jmath\to+\infty} \omega^p_\lambda(\delta_{m(\kappa)}, \delta_{\jmath(\kappa)}) \leq 0 \Rightarrow b = 0,$$

which is a contradiction. Thus $\{\delta_m\}$ is a Cauchy sequence in the complete modular space $\mathcal{N}_{\omega^s}$. By Lemma 2.2, $\{\delta_m\}$ is Cauchy sequence in the complete PMMS $(\mathcal{N}_{\omega^p}, \omega^p_\lambda)$. Hence there exists some $\imath \in \mathcal{N}_{\omega^p}$ such that

$$\lim_{\jmath,m\to+\infty} \omega^p_\lambda(\delta_m, \delta_n) = \lim_{m\to+\infty} \omega^p_\lambda(\delta_m, \imath) = \omega^p_\lambda(\imath, \imath) = 0, \text{ and}$$
$$\lim_{\jmath\to+\infty} \omega^p_\lambda(\mathcal{S}\delta_{2\jmath+1}, \imath) = \lim_{\jmath\to+\infty} \omega^p_\lambda(\mathcal{T}\delta_{2\jmath}, \imath) = 0.$$

Now, let $\omega^p_\lambda(\mathcal{T}\imath, \mathcal{S}\delta_{2\jmath+2}) \geq 0$,

$$\mathcal{F}(\omega^p_\lambda(\mathcal{T}\imath, \mathcal{S}\delta_{2\jmath+1})) \leq \mathcal{F}(\mathcal{G}(\imath, \delta_{2\jmath+1})) - \psi(\mathcal{G}(\imath, \delta_{2\jmath+1})), \tag{10}$$

where

$$\begin{aligned}\lim_{\jmath\to+\infty} \mathcal{G}(\imath, \delta_{2\jmath+1}) = \lim_{\jmath\to+\infty} & \max\{\omega^p_\lambda(\imath, \delta_{2\jmath+1}), \omega^p_\lambda(\imath, \mathcal{T}\imath), \omega^p_\lambda(\delta_{2\jmath+1}, \mathcal{S}\delta_{2\jmath+1}), \\ & \frac{[\omega^p_{2\lambda}(\imath, \mathcal{S}\delta_{2\jmath+1}) + \omega^p_{2\lambda}(\delta_{2\jmath+1}, \mathcal{T}\imath)]}{2}\} \\ = & \ \omega^p_\lambda(\mathcal{T}\imath, \imath)\end{aligned}$$

As $n \to +\infty$, from (10), we get

$$\mathcal{F}(\omega^p_\lambda(\mathcal{T}\imath, \imath)) \leq \mathcal{F}(\omega^p_\lambda(\mathcal{T}\imath, \imath)) - \psi(\omega^p_\lambda(\mathcal{T}\imath, \imath)),$$

As ψ is positive, hence a contradiction. So, $\omega^p_\lambda(\mathcal{T}\imath, \imath) = 0$.
Similarly, we can demonstrate that $\omega^p_\lambda(\mathcal{S}\imath, \imath) = 0$. Therefore, $\imath$ is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$.
Uniqueness: Let there exists $\imath^* \in \delta_{\omega^p}$ *such that* $\mathcal{S}\imath^* = \mathcal{T}\imath^* = \imath^*$.

$$\mathcal{F}(\omega^p_\lambda(\imath, \imath^*)) \leq \mathcal{F}(\mathcal{G}(\imath, \imath^*)) - \psi(\mathcal{G}(\imath, \imath^*)),$$

where

$$\begin{aligned}\mathcal{G}(\imath, \imath^*) = & \max\{\omega^p_\lambda(\imath, \imath^*), \omega^p_\lambda(\imath, P\imath), \omega^p_\lambda(\imath^*, Q\imath^*), \\ & \frac{[\omega^p_{2\lambda}(\imath, Q\imath^*) + \omega^p_{2\lambda}(\imath^*, P\imath)]}{2}\} \\ = & \ \omega^p_\lambda(\imath, \imath^*),\end{aligned}$$

$$\mathcal{F}(\omega_\lambda(\imath, \imath^*)) \leq \mathcal{F}(\omega^p_\lambda(\imath, \imath^*)) - \psi(\omega^p_\lambda(\imath, \imath^*)),$$

which is a contradiction, as ψ is positive. So, $\imath = \imath^*$. □

If in the above theorem, $\mathcal{S} = \mathcal{T}$, then we have the following result:

Theorem 3.2. *Let $(\mathcal{N}_{\omega^p}, \omega^p_\lambda)$ be a complete PMMS and $\mathcal{S} : \mathcal{N}_{\omega^p} \to \mathcal{N}_{\omega^p}$ be self-mappings satisfies $(\psi, \mathcal{GF})$-contraction mapping. Then $\mathcal{S}$ has a unique fixed point.*

Example 3.1. *Let $(\mathcal{N}_{\omega^p}, \omega^p_\lambda)$ be a PMMS, where $\mathcal{N}_{\omega^p} = [0,1]$, and $\omega^p_\lambda(\varrho_1, \varrho_2) = \frac{1}{\lambda}|\varrho_1 - \varrho_2| + \max\{\varrho_1, \varrho_2\}$, $\forall\lambda > 0$, $\varrho_1, \varrho_2 \in \mathcal{N}_{\omega^p}$.*
Let $\mathcal{T}$ and $\mathcal{S}$ denote self-mappings on $\mathcal{N}_{\omega^p}$ such that $\mathcal{T}\varrho_1 = \frac{\varrho_1}{4}$ and $\mathcal{S}\varrho_1 = \frac{\varrho_1}{2}$.
Let $\mathcal{F} : [0, +\infty) \to [0, +\infty)$ define by $\mathcal{F}(T) = T^2$; and $\psi : [0, +\infty) \to [0, +\infty)$ define by $\psi(T) = \frac{T^2}{2}$.

(I) *Clearly, $\mathcal{T}(\mathcal{N}_{\omega^p}) \subseteq \mathcal{S}(\mathcal{N}_{\omega^p})$.*

(II) *Moreover,* $\omega^p_\lambda(\mathcal{T}\varrho_1, \mathcal{S}\varrho_2)) \geq 0$, *and*

$$\begin{aligned}
\mathcal{F}(\omega^p_\lambda(\mathcal{T}\varrho_1, \mathcal{S}\varrho_2)) = [\omega^p_\lambda(\mathcal{T}\varrho_1, \mathcal{S}\varrho_2)]^2 &= [\omega^p_\lambda(\frac{\varrho_1}{4}, \frac{\varrho_2}{2})]^2 \\
&= [\frac{1}{\lambda}|\frac{\varrho_1}{4} - \frac{\varrho_2}{2}| + \max\{\frac{\varrho_1}{4}, \frac{\varrho_2}{2}\}]^2 \\
&\leq \frac{1}{2}[\frac{1}{\lambda}|\varrho_1 - \varrho_2| + \max\{\varrho_1, \varrho_2\}]^2 \\
&= \frac{1}{2}[\omega^p_\lambda(\varrho_1, \varrho_2)]^2 \\
&\leq \frac{1}{2}[\mathcal{G}(\varrho_1, \varrho_2)]^2 \\
&= [\mathcal{G}(\varrho_1, \varrho_2)]^2 - \frac{1}{2}[\mathcal{G}(\varrho_1, \varrho_2)]^2 \\
&= \mathcal{F}(\mathcal{G}(\varrho_1, \varrho_2)) - \psi(\mathcal{G}(\varrho_1, \varrho_2))
\end{aligned}$$

$$\Rightarrow \mathcal{F}(\omega^p_\lambda(\mathcal{T}\varrho_1, \mathcal{S}\varrho_2)) + \psi(\mathcal{G}(\varrho_1, \varrho_2)) \leq \mathcal{F}(\mathcal{G}(\varrho_1, \varrho_2)),$$

where $\mathcal{G}(\varrho_1, \varrho_2) = \max\{\omega^p_\lambda(\varrho_1, \varrho_2), \omega^p_\lambda(\varrho_1, \mathcal{T}\varrho_1), \omega^p_\lambda(\varrho_2, \mathcal{S}\varrho_2), \frac{[\omega^p_{2\lambda}(\varrho_2, \mathcal{T}\varrho_1) + \omega^p_{2\lambda}(\varrho_1, \mathcal{S}\varrho_2)]}{2}\}$.
So by theorem 3.1 $\mathcal{T}$ *and* $\mathcal{S}$ *has a unique fixed point* $\{0\}$.

4. Application

Consider the set of $\mathcal{V}$olterra type integral equations: ([8], [17])

(i)

$$\begin{aligned}
x(\varsigma) &= \int_0^\varsigma B_1(\varsigma, v, x(v))\, dv + h(\varsigma), \\
x(\varsigma) &= \int_0^\varsigma B_2(\varsigma, v, x(v))\, dv + h(\varsigma),
\end{aligned} \tag{11}$$

where $\varsigma \in [0, \wp] = \mathfrak{J} \subset \mathbb{R}$; $B_\Bbbk : [0, \wp] \times [0, \wp] \times \mathbb{R} \to \mathbb{R}$, $\Bbbk = \{1, 2\}$ and $h : [0, \wp] \to \mathbb{R}$ denote continuous functions.

(ii) Let $\mathrm{C}(\mathfrak{J}, \mathbb{R})$ denote the continuous functions space which is specified on $\mathfrak{J}$ and $\mathcal{S}, \mathcal{T} : \mathrm{C}(\mathfrak{J}, \mathbb{R}) \to \mathrm{C}(\mathfrak{J}, \mathbb{R})$ be self-mappings specified by:

$$\begin{aligned}
\mathcal{T}r(\varsigma) &= \int_0^\varsigma B_1(\varsigma, v, r(v))\, dv + h(\varsigma), \\
\mathcal{S}r(\varsigma) &= \int_0^\varsigma B_2(\varsigma, v, r(v))\, dv + h(\varsigma),
\end{aligned} \tag{12}$$

for all $v \in \mathrm{C}(\mathfrak{J}, \mathbb{R}), \varsigma \in \mathfrak{J}$. Clearly $r(\varsigma)$ is a solution of (i) if and only if it is a fixed point of $\mathcal{S}$ and $\mathcal{T}$.

(iii) Let $\mathcal{N}_{\omega^p} = \mathrm{C}(\mathfrak{J}, \mathbb{R})$ denote the space of continuous functions which is specified on $\mathfrak{J}$. Now, ω^p on $\mathcal{N}$ is specified by $\omega^p_\lambda(\varrho_1, \varrho_2) = \max_{\varsigma \in [0, \wp]}[e^{-\lambda}|\varrho_1(\varsigma) - \varrho_2(\varsigma)| + \max\{|\varrho_1(\varsigma)|, |\varrho_2(\varsigma)|\}]$, for all $\lambda > 0$.

Theorem 4.1. *Assume following hypotheses holds:*
There is a continuous function $\phi : \mathfrak{J} \times \mathfrak{J} \to \mathbb{R}^+$ *such that for all* $\varsigma, v \in \mathfrak{J}$ *and* $r, z \in \mathrm{C}(\mathfrak{J}, \mathbb{R})$.

(i)

$$\begin{aligned}
|B_1(\varsigma, v, r(v)) - B_2(\varsigma, v, z(v))| \leq \phi(\varsigma, v)[&|r(v) - z(v)| + e^\lambda \max\{|r(v)|, |z(v)|\} \\
&- \sqrt{2}e^\lambda \max\{|\mathcal{T}r(v)|, |\mathcal{S}z(v)|\}], \qquad \textit{where } \lambda > 0.
\end{aligned}$$

(ii) $\int_0^k |\phi(\varsigma, v)|\, dv \leq \frac{1}{\sqrt{2}}$.

Then, the integral equation (12) has a unique common solution in $\mathrm{C}(\mathfrak{J}, \mathbb{R})$.

Proof: Now,

$$\begin{aligned}
\omega_\lambda^p(\mathcal{T}r(\varsigma), \mathcal{S}z(\varsigma)) &= \max_{\varsigma\in[0,\wp]}[e^{-\lambda}|\mathcal{T}r(\varsigma) - \mathcal{S}z(\varsigma)| + \max\{|\mathcal{T}r(\varsigma)|, |\mathcal{S}z(\varsigma)|\}] \\
&= \max_{\varsigma\in[0,\wp]}[e^{-\lambda}\int_0^\varsigma |B_1(\varsigma, v, r(v)) - B_2(\varsigma, v, z(v))|dv + \max\{|\mathcal{T}r(\varsigma)|, |\mathcal{S}z(\varsigma)|\}] \\
&\leq \max_{\varsigma\in[0,\wp]}[e^{-\lambda}\int_0^\varsigma \phi(\varsigma, v)[|r(v) - z(v)| + e^\lambda \max\{|r(v)|, |z(v)|\} \\
&\quad - \sqrt{2}e^\lambda \max\{|\mathcal{T}r(v)|, |\mathcal{S}z(v)|\}]dv + \max\{|\mathcal{T}r(\varsigma)|, |\mathcal{S}z(\varsigma)|\}] \\
&\leq \frac{1}{\sqrt{2}}\omega_\lambda^p(r(\varsigma), z(\varsigma)).
\end{aligned}$$

Let $\psi : [0, +\infty) \to [0, +\infty)$ define by $\psi(P) = \frac{P^2}{2}$ and $\mathcal{F} : [0, +\infty) \to [0, +\infty)$ specified by $\mathcal{F} = P^2$.
Moreover, $\omega_\lambda^p(\mathcal{T}r, \mathcal{S}z)) \geq 0$, and

$$\mathcal{F}(\omega_\lambda^p(\mathcal{T}r, \mathcal{S}z)) + \psi(\omega_\lambda^p(\mathcal{G}(r, z)) \leq \mathcal{F}(\omega_\lambda^p(\mathcal{G}(r, z))),$$

where $\mathcal{G}(r, z) = \max\{\omega_\lambda^p(r, z), \omega_\lambda^p(r, \mathcal{T}r), \omega_\lambda^p(z, \mathcal{S}z), \frac{[\omega_{2\lambda}^p(z,\mathcal{T}r)+\omega_{2\lambda}^p(r,\mathcal{S}z)]}{2}\}$.
So, by theorem (3.1) we have a unique common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Hence, the system (12) of integral equations has unique solution in $\mathrm{C}(\mathfrak{J}, \mathbb{R})$.

Consider a fractional differential equation:(see [1], [2], [5], [9])

$$\begin{aligned}
&\mathfrak{D}^\alpha r(\varsigma) + \mathfrak{G}(\varsigma, r(\varsigma)) = 0,\ \varsigma \in \mathfrak{J} = [0, 1],\ 1 < \alpha \leq 2, \\
&r(0) = 0 = r(1)
\end{aligned} \tag{13}$$

where $\mathfrak{D}^\alpha$ represents the $\mathcal{C}$aputo $\mathfrak{f}$ractional $\mathfrak{d}$erivative of order α, $r \in \mathrm{C}(\mathfrak{J}, \mathbb{R}) = \mathcal{N}_{\omega^p}$, and $\mathfrak{G} : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ denote a continuous mapping.

Theorem 4.2. *Let $\delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given mapping and $\mathfrak{G} : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ denote a continuous mapping. Assume the following hypotheses holds:*

(i) *there exists $r_0, r \in \mathrm{C}(\mathfrak{J}, \mathbb{R})$ such that $\delta(r_0(\varsigma), \mathcal{S}r_0(\varsigma)) \geq 0$ for all $\varsigma, \varepsilon \in \mathfrak{J}$, where $\mathcal{S} : \mathrm{C}(\mathfrak{J}, \mathbb{R}) \to \mathrm{C}(\mathfrak{J}, \mathbb{R})$ is defined by*

$$\begin{aligned}
\mathcal{S}r(\varsigma) = \frac{1}{\Gamma(\beta)}\int_0^\varsigma (\varsigma(1-\varepsilon)^{(\beta-1)} - (\varsigma-\varepsilon)^{(\beta-1)})\mathfrak{G}(\varepsilon, r(\varepsilon))d\varepsilon \\
+ \frac{\varsigma}{\Gamma(\beta)}\int_\varsigma^1 (1-\varepsilon)^{(\beta-1)}\mathfrak{G}(\varepsilon, r(\varepsilon))d\varepsilon.
\end{aligned} \tag{14}$$

(ii) *for all $r, z \in \mathrm{C}(\mathfrak{J}, \mathbb{R})$, $\lambda > 0$,*

$$\begin{aligned}
|\mathfrak{G}(\varepsilon, r(\varepsilon)) - \mathfrak{G}(\varepsilon, z(\varepsilon))| \leq \frac{\Gamma(\beta+1)}{5\sqrt{2}}[|r(\varepsilon) - z(\varepsilon)| + e^\lambda \max\{|r(\varepsilon)|, |z(\varepsilon)|\} \\
- \sqrt{2}e^\lambda \max\{|\mathcal{S}r(\varepsilon)|, |\mathcal{S}z(\varepsilon)|\}].
\end{aligned}$$

(iii) *for all $\varsigma \in \mathfrak{J}$ and $r, z \in \mathrm{C}(\mathfrak{J}, \mathbb{R})$; $\delta(r(\varsigma), z(\varsigma)) \geq 0$ yields $\delta(\mathcal{S}r(\varsigma), \mathcal{S}z(\varsigma)) \geq 0$.*

(iv) *if $\{\varrho_n : n \in \mathbb{N}\} \in \mathrm{C}(\mathfrak{J}, \mathbb{R})$ is a sequence such that $\varrho_n \to x$ as $n \to +\infty$ then $\delta(\varrho_n, \varrho_{\jmath+1}) \geq 0$ yields $\delta(\varrho_n, x) \geq 0\ \forall\ n \in \mathbb{N}$.*

Then the system (14) of fractional differential equations with boundary conditions has solutions in $\mathrm{C}(\mathfrak{J},\mathbb{R})$.

Proof. Here, $r(\varsigma)\in \mathrm{C}(\mathfrak{J},\mathbb{R})$ is a solution of equation (13), if and only if it is the solution of the following integral equation

$$r(\varsigma)=\frac{1}{\Gamma(\beta)}\int_0^{\varsigma}(\varsigma(1-\varepsilon)^{(\beta-1)}-(\varsigma-\varepsilon)^{(\beta-1)})\mathfrak{G}(\varepsilon,r(\varepsilon))d\varepsilon + \frac{\varsigma}{\Gamma(\beta)}\int_{\varsigma}^{1}(1-\varepsilon)^{(\beta-1)}\mathfrak{G}(\varepsilon,r(\varepsilon))d\varepsilon.$$

So, a solution of this equation (13) is a fixed point of equation (14). Let $r,z\in \mathrm{C}(\mathfrak{J},\mathbb{R})$ such that $\delta(r(\varsigma),z(\varsigma))\geq 0$ for all $\varsigma\in\mathfrak{J}$. Now,

$$\begin{aligned}
\omega^p_\lambda(\mathcal{S}r(\varsigma),\mathcal{S}z(\varsigma)) &= \max_{\varsigma\in[0,1]}\Big[e^{-\lambda}|\mathcal{S}r(\varsigma)-\mathcal{S}z(\varsigma)|+\max\{|\mathcal{S}r(\varsigma)|,|\mathcal{S}z(\varsigma)|\}\Big]\\
&\leq \max_{\varsigma\in[0,1]}\Big[\frac{e^{-\lambda}}{\Gamma(\beta)}\int_0^{\varsigma}(\varsigma(1-\varepsilon)^{(\beta-1)}-(\varsigma-\varepsilon)^{(\beta-1)})|\mathfrak{G}(\varepsilon,r(\varepsilon))-\mathfrak{G}(\varepsilon,z(\varepsilon))|d\varepsilon\\
&\qquad + e^{-\lambda}\frac{\varsigma}{\Gamma(\beta)}\int_{\varsigma}^{1}(1-\varepsilon)^{(\beta-1)}|\mathfrak{G}(\varepsilon,r(\varepsilon))-\mathfrak{G}(\varepsilon,z(\varepsilon))|d\varepsilon\\
&\qquad + \max\{|\mathcal{S}r(\varsigma)|,|\mathcal{S}z(\varsigma)|\}\Big]\\
&\leq \max_{\varsigma\in[0,1]}\Big[\frac{\Gamma(\beta+1)}{5\sqrt{2}}\{e^{-\lambda}|r(\varepsilon)-z(\varepsilon)|+\max\{|r(\varepsilon)|,|z(\varepsilon)|\}\\
&\qquad -\sqrt{2}\max\{|\mathcal{S}r(\varepsilon)|,|\mathcal{S}z(\varepsilon)|\}\}\\
&\quad\times\{\frac{1}{\Gamma(\beta)}\int_0^{\varsigma}(\varsigma(1-\varepsilon)^{(\beta-1)}-(\varsigma-\varepsilon)^{(\beta-1)})d\varepsilon+\frac{\varsigma}{\Gamma(\beta)}\int_{\varsigma}^{1}(1-\varepsilon)^{(\beta-1)}d\varepsilon\}\\
&\qquad + \max\{|\mathcal{S}r(\varsigma)|,|\mathcal{S}z(\varsigma)|\}\Big]\\
&\leq \frac{1}{\sqrt{2}}\max_{\varsigma\in[0,1]}[e^{-\lambda}|r(\varepsilon)-z(\varepsilon)|+\max\{|r(\varepsilon)|,|z(\varepsilon)|\}]\\
&= \frac{1}{\sqrt{2}}\omega^p_\lambda(r(\varsigma),z(\varsigma)).
\end{aligned}$$

Let $\psi:[0,+\infty)\to[0,+\infty)$ define by $\psi(P)=\frac{P^2}{2}$ and $\mathcal{F}:[0,+\infty)\to[0,+\infty)$ define by $\mathcal{F}(P)=P^2$.
Moreover, $\omega^p_\lambda(\mathcal{S}r,\mathcal{S}z)\geq 0$, and

$$\mathcal{F}(\omega^p_\lambda(\mathcal{S}r,\mathcal{S}z))+\psi(\omega^p_\lambda(\mathcal{G}(r,z))\leq \mathcal{F}(\omega^p_\lambda(\mathcal{G}(r,z)),$$

where $\mathcal{G}(r,z)=\max\{\omega^p_\lambda(r,z),\omega^p_\lambda(r,\mathcal{S}r),\omega^p_\lambda(z,\mathcal{S}z),\frac{[\omega^p_{2\lambda}(z,\mathcal{S}r)+\omega^p_{2\lambda}(r,\mathcal{S}z)]}{2}\}$.
So, by theorem (3.2) we have a unique fixed point of $\mathcal{S}$. □

5. Concluding Remarks:

Within this study, we show some common fixed-point theorems employing $(\psi,\mathcal{GF})$-contraction condition to obtain possible generalized fixed point results. Due to the huge potential for application in a various field of science and engineering, we discuss applications for fractional differential equations and $\mathcal{V}$olterra integral equations. As PMMS is a generalized metric space, studying fixed point results in such spaces generalizes existing results given in the literature. The structure of this kind of metric space may help in the progress of PMS in different aspects based on modular.

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