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# INCORPORATION OF $\alpha^{\mathfrak{M}}$ CLOSED SETS OF NEUTROSOPHIC OVER SOFT MODAL TOPOLOGICAL STRUCTURES FOR REVOLUTIONIZING HOSPITAL RANKING WITH NUMERICAL INSIGHTS

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Abstract. The significance of Neutrosophic environments continues to grow across various disciplines through ongoing advancements. A notable recent development is the Neutrosophic Over Soft Modal Topological Structure ($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topology), which integrates Neutrosophic topological operations ⊞ (closure) and ⊡ (interior) with modal operations ⊛ (modal closure) and ⊙ (modal interior). This work introduces two such structures for the first time, detailing their core properties and addressing challenges related to uncertainty and indeterminacy. Key concepts developed include Neutrosophic Over Soft Modal $\alpha$ closed, $\alpha g$ closed, and $\alpha^{\mathfrak{m}}$ closed sets, along with various forms of continuity such as modal $\alpha^{\mathfrak{m}}$ and strong continuity. A real-life application is presented for selecting the best cancer hospital using cosine correlation on Neutrosophic Over Sets, demonstrating the practicality and strength of the proposed framework.

Keywords: Neutrosophic-over-soft-modal topological structure ($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological structure), neutrosophic over soft modal $\alpha$-closed set, neutrosophic over soft modal $g$-closed set, neutrosophic over soft modal $\alpha^{\mathfrak{m}}$-closed set, neutrosophic over soft modal continuous function, strong neutrosophic over soft modal $\alpha^{\mathfrak{m}}$ continuous function.

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## 1. Introduction

In recent years, increasing attention has been given to the question of decision-making in such environments because uncertainty, and imprecision are said to lie outside the purview of traditional theories. The concept of fuzzy sets by Zadeh in 1965 provides a mechanism for dealing with the uncertainties of partial membership into a set [24]. Zadeh subsequently extended this work by elaborating on possibility theory which gives additional leeways in managing uncertainty [25]. Following these concepts, Bellman and Zadeh in the late 70s and early 80s sought to establish a coherent framework for decision-making in what is now called fuzzy environments, which has since been applied extensively across various spectra of influence [3].

In the 1980s, Atanassov refined Zadeh's concept through introducing intuitionistic fuzzy sets, which have incorporated two additional input parameters: membership and non-membership degrees, thus providing an improved framework for modeling uncertain information [1]. This approach has also been refined and put into use in many areas including the larger framework of modal topological structure as mentioned above [2]. Indeed, Bustince and Burillo contributed to this with respect to correlation measures within interval-valued intuitionistic fuzzy sets, thereby giving better decision-making strategies for systems inherently fraught with interval-based uncertainties [5].

Molodtsov (1999) [16] introduced soft set theory as a new mathematical approach to handle uncertainties without the limitations of traditional methods. This foundational work laid the groundwork for numerous applications in decision-making and data analysis. Introduction to soft-set theory was this update by Maji et al. with a view to employ this tool decisively in solving multi-criteria decision-making problems[14].Smarandache (1999) [20] proposed neutrosophic logic as a general framework that unifies and extends classical, fuzzy, and intuitionistic logic. This theory allows simultaneous representation of truth, indeterminacy, and falsity, enabling advanced reasoning under uncertainty.Smarandache (2016) [21] introduced the concepts of neutrosophic overset, underset, and offset to extend neutrosophic logic, allowing components to exceed classical bounds.

[11] Dhavaseelan et al. (2019) proposed neutrosophic alpha m-continuity as an extension to the existing neutrosophic topology. Such a work offers perspectives and applications in the area. Applications of neutrosophic sets have been in the areas of medical diagnostics, decision-making, cognitive mapping.

Wang and colleagues, within the purview of neutrosophic theory,[22] introduced single-valued neutrosophic sets that extend the very basic neutrosophic concepts into more practically applicable dimensions. Subsequently, this extension worked successfully in decision support systems and different real-life situations. On such important developments, Ye proposed the new correlation coefficients for single-valued neutrosophoric sets and techniques for multi-attribute decision analysis [23].

Neutrosophic sets ($\mathcal{NT}$) abstract fuzzy sets and allow much more for access to different truth, falsehood, and both nondeterministic and obscure values, on which pythagorean neutrosophic sets (PNS) refer but gives much more varied. This was well expounded by someone like Jansi et al., where they discussed correlation measures for PNS with dependent components [12]. By the same token, Radha et al. strived extremely hard to improve PNS correlation coefficients to realize higher precision in uncertain decision environments [18].

Devi and Parthiban's work has been of significance, where they have stated the applicability of Pythagorean neutrosophic soft set in various areas of the decision-making process like the health-care sector [9], soft topology study [7], school selection using TOPSIS method [8], and chemistry as a decision field[10]. They have done excellent work giving very sound theoretical backgrounds on using neutrosophic sets in decision-making models.

Further, for theoretical propositions, Mallick and Pramanik have suggested that pentapartitioned neutrosophic sets give opportunities for neutrosophic logic to tackle more complex problems [15]. The other situation would be application of neutrosophic cognitive map to analyse the behaviour of dengue fever [13] and COVID-19 variants[17], suggesting that applications of neutrosophic oddities continue to grow in applicability in real-world problem-solving.

Other extended applications Neutrosophic Logic, according to Christianto, Boyd, and Smarandache, are basic and applied sciences that show the developing cross-disciplines of the subject [6]. Moreover, Broumi's developments on general neutrosophic soft sets can also be included [4], which define expanded utilizations of neutrosophy into included domains and fields.

The theoretical foundation on which Neutrosophic Over Soft Modal Topological Structure ($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological structure) is based has been explained within this manuscript. It illustrates the conceptual understanding of such a framework. The $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological structure would thus imply that through the operations $\boxplus$ (closure) and $\boxdot$ (interior) from traditional Neutrosophic Over topological models, two distinct forms can be produced through standard Neutrosophic Over Soft Modal operations: $\circledast$ (modal closure) and $\odot$ (modal interior). These operations fill in the gaps in the literature currently available and address applications of those existing models that cannot handle the indeterminacies, uncertainties, and contradictions existent in realities. More so, this research explores higher innovations such as Neutrosophic Over Soft Modal $\alpha$ closed set, Neutrosophic Over Soft Modal $\alpha g$ closed set, Neutrosophic Over Soft Modal $\alpha^{\mathfrak{m}}$ closed set, Neutrosophic Over Continuous Function, Neutrosophic Over Soft Modal $\alpha^m$ Continuous Function, and Strong Neutrosophic Over Soft Modal $\alpha^m$ Continuous Function. Specifically, the modal closure and modal interior operations refine the greater nuances found in decision-making, thus improving processes regarding modal dynamic systems in uncertain environments. In combining both theoretical and applied aspects of the field, this research presents the topic comprehensively to the interested theoretical researcher or practitioner. Such framework application would thus be recommended for real-worl scenarios under uncertainty, contradiction, and indeterminacy by deploying possible solutions through cosine correlation within the neutrosophic over environment.

## 2. Preliminary

This is the necessary description of the fundamental concepts and definitions from which the understanding is built; these are Neutrosophic Over Soft Set[7] and Neutrosophic Over Soft Topological Spaces[7].

**Definition 2.1.** [7] *Let $\mathcal{M}$ be an non-empty set and $\mathcal{E}$ be a set of parameter on $\mathcal{M}$. Then $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set is defined by a set valued function*

$$\lambda_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}} : \mathcal{E} \to \rho(\mathcal{M})$$

*where $\rho(\mathcal{M})$ is an set of all $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set on $\mathcal{M}$. $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set is an valued function from the set of parameter $\mathcal{E}$ on $\mathcal{M}$ is defined as*

$$\mathcal{T} = (\lambda_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}, \mathcal{E}) = \{(\mathsf{e}, \{\langle \mathsf{h}, \aleph_{\mathcal{T}}(\mathsf{h}), \eth_{\mathcal{T}}(\mathsf{h}), \Upsilon_{\mathcal{T}}(\mathsf{h})\rangle : \mathsf{h} \in \mathcal{M}\}) : \mathsf{e} \in \mathcal{E}\}$$

**Definition 2.2.** [7] *A $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\triangleright = \{e, \{\langle \mathsf{h}, 0, 0, \Omega\rangle : \mathsf{h} \in \mathcal{M}\} : e \in \mathcal{E}\}$ is said to be a Null $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set and $\blacktriangleright = \{e, \{\langle \mathsf{h}, \Omega, \Omega, 0\rangle : \mathsf{h} \in \mathcal{M}\} : e \in \mathcal{E}\}$ is said to be an universal $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set.*

**Definition 2.3.** [7] *A neutrosophic over soft topology($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topology) $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ on non-empty set $\mathcal{M}$ such that*

*(i) $\triangleright, \blacktriangleright \in \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$.*

*(ii) The union of an arbitrary collection $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ is in $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$.*

*(iii) The finite intersection of subsets $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ is in $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$.*

*Then $(\mathcal{M}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is called neutrosophic over soft topological space($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space).An element of $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ is called an neutrosophic over soft open set($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}-$openset) and complement of $\tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$ neutrosophic over soft closed set($\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}-$closedset).*

**Definition 2.4.** [7] *An operators of $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\mathcal{T} \in \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}$, then neutrosophic over soft topological closure and interior are $cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})$ and $int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})$ is defined as:*

$cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T}) = \Omega\{\mathcal{G} : \mathcal{G} \quad is \quad \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} - closedset \quad in \quad \mathcal{M} \quad and \quad \mathcal{T} \subseteq \mathcal{G}\}$.

$int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T}) = \mho\{\mathcal{O} : \mathcal{O} \quad is \quad \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} - openset \quad in \quad \mathcal{M} \quad and \quad \mathcal{T} \supseteq \mathcal{O}\}$.

**Note:** In this paper, the Neutrosophic Over Soft Set
*mathcalT* is defined initially as

$$\mathcal{T} = (\lambda_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}, \mathcal{E}),$$

in terms of elements $\mathsf{e} \in \mathcal{E}$ and corresponding maps involving $\mathsf{h} \in \mathcal{M}$. Yet, in the main work throughout, $\mathcal{T}$ is always represented as

$$\mathcal{T} = \{\langle \mathsf{h}, \aleph_{\mathcal{T}}(\mathsf{h}), \eth_{\mathcal{T}}(\mathsf{h}), \Upsilon_{\mathcal{T}}(\mathsf{h})\rangle : \mathsf{h} \in \mathcal{M}\},$$

emphasizing the basic relations within the framework.

## 3. Neutrosophic Over Soft Modal Topological Structure

Detailed description of Neutrosophic Over Soft Modal Topological Structure were discussed.

**Definition 3.1.** *Let $\mathcal{T}$ be an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set in $\mathcal{M}$.Then two standard $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal closure and $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal interior operations are expressed as $\circledast$ and $\odot$ is defined as*

$$\circledast(\mathcal{T}) = \{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Omega - \sqrt{|\aleph_{\mathcal{T}}(\mathfrak{h}) - \eth_{\mathcal{T}}(\mathfrak{h})|}\rangle : \mathfrak{h} \in \mathcal{M}\}$$

$$\odot\mathcal{T} = \{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \sqrt{(\aleph_{\mathcal{T}}(\mathfrak{h}))(\Upsilon_{\mathcal{T}}(\mathfrak{h}))}, |\sqrt{\Omega} - \Upsilon_{\mathcal{T}}(\mathfrak{h})|\rangle : \mathfrak{h} \in \mathcal{M}\}$$

**Example 3.1.** *Let $\mathcal{M} = \{\mathfrak{h}\}$ be a non-empty set, $\Omega = 1.5$ be a over limit and $\mathcal{T} = \{\langle \mathfrak{h}, 1.3, 0.8, 0.7\rangle\}$ be an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set. Then,*

$$\circledast(\mathcal{T}) = \{\langle \mathfrak{h}, 1.3, 0.8, 0.5\rangle\} and \odot\mathcal{T} = \{\langle \mathfrak{h}, 1.3, 0.9, 0.5\rangle\}$$

**Remark 3.1.** *For an ordinary $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\mathcal{T}$, $\circledast(\mathcal{T}) \subseteq \mathcal{T} \subseteq \odot\mathcal{T}$*

**Definition 3.2.** *Let $\mathcal{M}$ is a non-empty set.*

(a) *A mapping $\circledast$: $\rho(\mathcal{M}) \to \rho(\mathcal{M})$ is an operator over $\mathcal{M}$ and $\mho, \Omega : \rho(\mathcal{M}) \times \rho(\mathcal{M}) \to \mathcal{M}^{\hbar}$ are operation over $\mathcal{M}$. Then*

(i) $cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T}\mho\mathcal{W}) = cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})\mho cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{W})$

(ii) $\mathcal{T} \subseteq cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})$

(iii) $cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\triangleright) = \blacktriangleright$

(iv) $cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})) = cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})$

(v) $\circledast(\mathcal{T}\Omega\mathcal{W}) = \circledast(\mathcal{T})\Omega \circledast(\mathcal{W})$

(vi) $\circledast(\mathcal{T}) \subseteq \mathcal{T}$

(vii) $\circledast(\triangleright) = \triangleright$

(viii) $\circledast(\circledast(\mathcal{T})) = \circledast(\mathcal{T})$

(ix) $\circledast\,(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})) = cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\circledast\,(\mathcal{T}))$

*For all* $\mathcal{T}, \mathcal{W} \in \rho(\mathcal{M})$. *A pair* $\langle \rho(\mathcal{M}), cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}, \mho, \sqcap, \circledast \rangle$ *is satisfies the above conditions (i)-(ix) it is known as Closure-Neutrosophic Over Soft Modal Topology(cl-*$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *Modal Topology).*

(b) *A mapping* $\odot : \rho(\mathcal{M}) \to \rho(\mathcal{M})$ *is an operator over* $\mathcal{M}$ *and* $\mho, \sqcap : \rho(\mathcal{M}) \times \rho(\mathcal{M}) \to \mathcal{M}^{\hbar}$ *are operation over* $\mathcal{M}$. *Then*

(i) $int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T} \sqcap \mathcal{W}) = int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T}) \sqcap int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{W})$

(ii) $int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T}) \subseteq \mathcal{T}$

(iii) $int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\blacktriangleright) = \triangleright$

(iv) $int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})) = int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})$

(v) $\odot(\mathcal{T} \mho \mathcal{W}) = \odot(\mathcal{T}) \mho \odot(\mathcal{W})$

(vi) $\mathcal{T} \subseteq \odot(\mathcal{T})$

(vii) $\odot(\triangleright) = \triangleright$

(viii) $\odot(\odot(\mathcal{T})) = \odot(\mathcal{T})$

(ix) $\odot(int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})) = int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\odot(\mathcal{T}))$

*For all* $\mathcal{T}, \mathcal{W} \in \rho(\mathcal{M})$. *A pair* $\langle \rho(\mathcal{M}), int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}, \sqcap, \mho, \odot \rangle$ *is satisfies the above conditions (i)-(ix) it is known as Interior-Neutrosophic Over Soft Modal Topology(int-*$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *Modal Topology).*

**Theorem 3.1.** *A pair* $\langle \rho(\mathcal{M}), cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}, \mho, \sqcap, \circledast \rangle$ *is cl-*$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *Modal Topology.*

*Proof.* Let $\mathcal{T}$ and $\mathcal{W}$ be two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-sets in $\rho(\mathcal{M})$

(i)
$$\begin{aligned}
cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T} \mho \mathcal{W}) &= cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Upsilon_{\mathcal{T}}(\mathfrak{h})\rangle : \mathfrak{h} \in \mathcal{M}\} \mho \\
&\qquad \{\langle \mathfrak{h}, \aleph_{\mathcal{W}}(\mathfrak{h}), \eth_{\mathcal{W}}(\mathfrak{h}), \Upsilon_{\mathcal{W}}(\mathfrak{h})\rangle : \mathfrak{h} \in \mathcal{M}\}) \\
&= cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\{\langle \mathfrak{h}, max(\aleph_{\mathcal{T}}(\mathfrak{h}), \aleph_{\mathcal{W}}(\mathfrak{h})), max(\eth_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{W}}(\mathfrak{h})), \\
&\qquad min(\Upsilon_{\mathcal{T}}(\mathfrak{h}), \Upsilon_{\mathcal{W}}(\mathfrak{h}))\rangle : \mathfrak{h} \in \mathcal{M}\}) \\
&= \{\langle \mathfrak{h}, \sup_{\mathsf{d} \in \mathcal{M}} max(\aleph_{\mathcal{T}}(\mathsf{d}), \aleph_{\mathcal{W}}(\mathsf{d})), \sup_{\mathsf{d} \in \mathcal{M}} max(\eth_{\mathcal{T}}(\mathsf{d}), \eth_{\mathcal{W}}(\mathsf{d})), \\
&\qquad \inf_{\mathsf{d} \in \mathcal{M}} min(\Upsilon_{\mathcal{T}}(\mathsf{d}), \Upsilon_{\mathcal{W}}(\mathsf{d}))\rangle : \mathfrak{h} \in \mathcal{M}\} \\
&= \{\langle \mathfrak{h}, max(\sup_{\mathsf{d} \in \mathcal{M}} \aleph_{\mathcal{T}}(\mathsf{d}), \sup_{\mathsf{d} \in \mathcal{M}} \aleph_{\mathcal{W}}(\mathsf{d})), \\
&\qquad max(\sup_{\mathsf{d} \in \mathcal{M}} \eth_{\mathcal{T}}(\mathsf{d}), \sup_{\mathsf{d} \in \mathcal{M}} \eth_{\mathcal{W}}(\mathsf{d})), \\
&\qquad min(\inf_{\mathsf{d} \in \mathcal{M}} \Upsilon_{\mathcal{T}}(\mathsf{d}), \inf_{\mathsf{d} \in \mathcal{M}} \Upsilon_{\mathcal{W}}(\mathsf{d}))\rangle : \mathfrak{h} \in \mathcal{M}\}
\end{aligned}$$
$\therefore cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T} \mho \mathcal{W}) = cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T}) \mho cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{W})$.

(ii)
$$\begin{aligned}
\mathcal{T} = \{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Upsilon_{\mathcal{T}}(\mathfrak{h})\rangle : \mathfrak{h} \in \mathcal{M}\} &\subseteq \{\langle \mathfrak{h}, \sup_{\mathsf{d} \in \mathcal{M}} \aleph_{\mathcal{T}}(\mathsf{d}), \sup_{\mathsf{d} \in \mathcal{M}} \eth_{\mathcal{T}}(\mathsf{d}), \\
&\qquad \inf_{\mathsf{d} \in \mathcal{M}} \Upsilon_{\mathcal{T}}(\mathsf{d})\rangle : \mathfrak{h} \in \mathcal{M}\}
\end{aligned}$$
$\therefore \mathcal{T} \subseteq cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})$

(iii)
$$\begin{aligned}
cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\triangleright) &= cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\{\langle \mathfrak{h}, 0, 0, \Omega\rangle : \mathfrak{h} \in \mathcal{M}\}) \\
&= \{\langle \mathfrak{h}, \sup_{\mathsf{d} \in \mathcal{M}} 0, \sup_{\mathsf{d} \in \mathcal{M}} 0, \inf_{\mathsf{d} \in \mathcal{M}} \Omega\rangle : \mathfrak{h} \in \mathcal{M}\}
\end{aligned}$$
$\therefore cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\triangleright) = \blacktriangleright$

(iv)
$$\begin{aligned}
cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})) &= cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Upsilon_{\mathcal{T}}(\mathfrak{h})\rangle : \mathfrak{h} \in \mathcal{M}\})) \\
&= cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\{\langle \mathfrak{h}, \sup_{\mathsf{d} \in \mathcal{M}} \aleph_{\mathcal{T}}(\mathsf{d}), \sup_{\mathsf{d} \in \mathcal{M}} \eth_{\mathcal{T}}(\mathsf{d}), \inf_{\mathsf{d} \in \mathcal{M}} \Upsilon_{\mathcal{T}}(\mathsf{d})\rangle : \mathfrak{h} \in \mathcal{M}\}) \\
&= \{\langle \mathfrak{h}, \sup_{\mathsf{d} \in \mathcal{M}} \aleph_{\mathcal{T}}(\mathsf{d}), \sup_{\mathsf{d} \in \mathcal{M}} \eth_{\mathcal{T}}(\mathsf{d}), \inf_{\mathsf{d} \in \mathcal{M}} \Upsilon_{\mathcal{T}}(\mathsf{d})\rangle : \mathfrak{h} \in \mathcal{M}\}
\end{aligned}$$
$\therefore cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})) = cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})$

(v)
$$\begin{aligned}
\circledast\,(\mathcal{T} \sqcap \mathcal{W}) &= \circledast\,(\{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Upsilon_{\mathcal{T}}(\mathfrak{h})\rangle : \mathfrak{h} \in \mathcal{M}\} \sqcap \{\langle \mathfrak{h}, \aleph_{\mathcal{W}}(\mathfrak{h}), \eth_{\mathcal{W}}(\mathfrak{h}), \\
&\qquad \Upsilon_{\mathcal{W}}(\mathfrak{h})\rangle : \mathfrak{h} \in \mathcal{M}\}) \\
&= \circledast\,(\{\langle \mathfrak{h}, min(\aleph_{\mathcal{T}}(\mathfrak{h}), \aleph_{\mathcal{W}}(\mathfrak{h})), min(\eth_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{W}}(\mathfrak{h})), max(\Upsilon_{\mathcal{T}}(\mathfrak{h}), \\
&\qquad \Upsilon_{\mathcal{W}}(\mathfrak{h}))\rangle : \mathfrak{h} \in \mathcal{M}\}) \\
&= \{\langle \mathfrak{h}, min(\aleph_{\mathcal{T}}(\mathfrak{h}), \aleph_{\mathcal{W}}(\mathfrak{h})), min(\eth_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{W}}(\mathfrak{h})), \\
&\qquad \Omega - \sqrt{|min(\aleph_{\mathcal{T}}(\mathfrak{h}), \aleph_{\mathcal{W}}(\mathfrak{h})) - min(\eth_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{W}}(\mathfrak{h}))|}\rangle : \mathfrak{h} \in \mathcal{M}\} \\
&= \{\langle \mathfrak{h}, min(\aleph_{\mathcal{T}}(\mathfrak{h}), \aleph_{\mathcal{W}}(\mathfrak{h})), min(\eth_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{W}}(\mathfrak{h})), \\
&\qquad \Omega - \sqrt{|max((|\aleph_{\mathcal{T}}(\mathfrak{h}) - \eth_{\mathcal{T}}(\mathfrak{h})|), (\aleph_{\mathcal{W}}(\mathfrak{h}) - \eth_{\mathcal{W}}(\mathfrak{h})))|}\rangle : \mathfrak{h} \in \mathcal{M}\}
\end{aligned}$$

$\therefore \circledast(\mathcal{T} \Omega \mathcal{W}) = \circledast(\mathcal{T}) \Omega \circledast(\mathcal{W})$

(vi) Proof is obvious from the remark 3.1

(vii) $\circledast(\rhd) = \circledast(\{\langle \mathfrak{h}, 0, 0, \Omega\rangle : \mathfrak{h} \in \mathcal{M}\})$
$= \{\langle \mathfrak{h}, 0, 0, \Omega\rangle : \mathfrak{h} \in \mathcal{M}\}$
$\therefore \circledast(\rhd) = \rhd$

(viii) $\circledast(\circledast(\mathcal{T})) = \circledast(\{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Omega - \sqrt{|\aleph_{\mathcal{T}}(\mathfrak{h}) - \eth_{\mathcal{T}}(\mathfrak{h})|}\rangle : \mathfrak{h} \in \mathcal{M}\})$
$= \{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Omega - \sqrt{|\aleph_{\mathcal{T}}(\mathfrak{h}) - \eth_{\mathcal{T}}(\mathfrak{h})|}\rangle : \mathfrak{h} \in \mathcal{M}\} = \circledast(\mathcal{T})$
$\therefore \circledast(\circledast(\mathcal{T})) = \circledast(\mathcal{T})$

(ix) $\circledast(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})) = \{\langle \mathfrak{h}, \sup_{\mathsf{d}\in\mathcal{M}} \aleph_{\mathcal{T}}(\mathsf{d}), \sup_{\mathsf{d}\in\mathcal{M}} \eth_{\mathcal{T}}(\mathsf{d}), \inf_{\mathsf{d}\in\mathcal{M}} \Upsilon_{\mathcal{T}}(\mathsf{d})\rangle : \mathfrak{h} \in \mathcal{M}\}$
$= \{\langle \mathfrak{h}, \sup_{\mathsf{d}\in\mathcal{M}} \aleph_{\mathcal{T}}(\mathsf{d}), \sup_{\mathsf{d}\in\mathcal{M}} \eth_{\mathcal{T}}(\mathsf{d}),$
$\Omega - \sqrt{|\sup_{\mathsf{d}\in\mathcal{M}} \aleph_{\mathcal{T}}(\mathsf{d}) - \sup_{\mathsf{d}\in\mathcal{M}} \eth_{\mathcal{T}}(\mathsf{d}|)}\rangle : \mathfrak{h} \in \mathcal{M}\}$
$= \{\langle \mathfrak{h}, \sup_{\mathsf{d}\in\mathcal{M}} \aleph_{\mathcal{T}}(\mathsf{d}), \sup_{\mathsf{d}\in\mathcal{M}} \eth_{\mathcal{T}}(\mathsf{d}),$
$\inf_{\mathsf{d}\in\mathcal{M}} \Omega - \sqrt{|\aleph_{\mathcal{T}}(\mathsf{d}) - \eth_{\mathcal{T}}(\mathsf{d})|}\rangle : \mathfrak{h} \in \mathcal{M}\}$
$= cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Omega - \sqrt{|\aleph_{\mathcal{T}}(\mathfrak{h}) - \eth_{\mathcal{T}}(\mathfrak{h})|}\rangle : \mathfrak{h} \in \mathcal{M}\})$
$\therefore \circledast(cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\mathcal{T})) = cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}(\circledast(\mathcal{T}))$

Hence $\langle \rho(\mathcal{M}), cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}, \mho, \Omega, \circledast\rangle$ is cl-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ Modal Topology. □

**Theorem 3.2.** *A pair $\langle \rho(\mathcal{M}), int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}, \Omega, \mho, \odot\rangle$ is int-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal topology.*

*Proof.* The proof similarly prove by the theorem 3.1. □

**Proposition 3.1.** *Let $\langle \rho(\mathcal{M}), cl_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}, \mho, \Omega, \circledast\rangle$ and $\langle \rho(\mathcal{M}), int_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}, \Omega, \mho, \odot\rangle$ be a cl-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal topology and int-$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal topology. For a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\mathcal{T}$ in $\mathcal{M}$, then*

(i) $\circledast(\mathcal{T})$ *is the smallest $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal closed set contained in $\mathcal{T}$.*
(ii) $\odot\mathcal{T}$ *is the largest $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal open set contains $\mathcal{T}$.*

*Proof.* (i) By (a) of definition 3.2,
$\circledast(\mathcal{T}) = \{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Omega - \sqrt{|\aleph_{\mathcal{T}}(\mathfrak{h}) - \eth_{\mathcal{T}}(\mathfrak{h})|}\rangle : \mathfrak{h} \in \mathcal{M}\}$
Then by the remark 3.1
$\circledast(\mathcal{T}) \subseteq \mathcal{T} \subseteq \odot\mathcal{T}$
$\Longrightarrow \circledast(\mathcal{T}) \subseteq \mathcal{T}$
$\therefore$ $\circledast(\mathcal{T})$ is the smallest $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal closed set contained in $\mathcal{T}$.

(ii) By (b) of definition 3.2,
$\odot\mathcal{T} = \{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \sqrt{(\aleph_{\mathcal{T}}(\mathfrak{h}))(\Upsilon_{\mathcal{T}}(\mathfrak{h}))}, |\sqrt{\Omega} - \Upsilon_{\mathcal{T}}(\mathfrak{h})|\rangle : \mathfrak{h} \in \mathcal{M}\}$
Then by the remark3.1
$\circledast(\mathcal{T}) \subseteq \mathcal{T}$
$\mathcal{T} \subseteq \odot\mathcal{T}$
$\Longrightarrow \mathcal{T} \subseteq \odot\mathcal{T}$
$\therefore$ $\odot\mathcal{T}$ is the largest $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal open set contains in $\mathcal{T}$. □

**Definition 3.3.** *Let two neutrosophic over soft sets as $\mathcal{T} = \{\langle \mathfrak{h}, \aleph_{\mathcal{T}}(\mathfrak{h}), \eth_{\mathcal{T}}(\mathfrak{h}), \Upsilon_{\mathcal{T}}(\mathfrak{h})\rangle : \mathfrak{h} \in \mathcal{M}\}$ and $\mathcal{W} = \{\langle \mathsf{j}, \aleph_{\mathcal{W}}(\mathsf{j}), \eth_{\mathcal{W}}(\mathsf{j}), \Upsilon_{\mathcal{W}}(\mathsf{j})\rangle : \mathsf{j} \in \mathcal{T}\}$ and $\mathfrak{f} : \mathcal{M} \to \mathcal{T}$ be a function,*

*(i)$\mathfrak{f}(\mathcal{T}) = \{\langle \mathsf{j}, \mathfrak{f}(\aleph_{\mathcal{T}})(\mathsf{j}), \mathfrak{f}(\eth_{\mathcal{T}})(\mathsf{j}), (1 - \mathfrak{f}(1 - \Upsilon_{\mathcal{T}}))(\mathsf{j})\rangle : \mathsf{j} \in \mathcal{T}\}$ is a neutrosophic over soft set on $\mathcal{T}$ called the image of $\mathcal{T}$ under $\mathfrak{f}$*

*(ii) $\mathfrak{f}^{\rightarrow}(\mathcal{W}) = \{\langle x, \mathfrak{f}^{\rightarrow}(\aleph_{\mathcal{W}})(\mathfrak{h}), \mathfrak{f}^{\rightarrow}(\eth_{\mathcal{W}})(\mathfrak{h}), \mathfrak{f}^{\rightarrow}(\Upsilon_{\mathcal{W}})(\mathfrak{h})\rangle : \mathfrak{h} \in \mathcal{M}\}$ is a neutrosophic over soft set on $\mathcal{W}$ is called the pre-image of $\mathcal{W}$ under $\mathfrak{f}$, where*

$$\mathfrak{f}(\Upsilon_{\mathcal{T}})(\mathfrak{j}) = \begin{cases} \sup_{\mathfrak{h}\in\mathfrak{f}^{\rightarrow}(\mathfrak{j})} \Upsilon_{\mathcal{T}}(\mathfrak{h}), & \textit{if } \mathfrak{f}^{\rightarrow}(\mathfrak{j}) \neq \textit{ptyset} \\ 0, & \textit{otherwise} \end{cases}$$

$$\mathfrak{f}(\eth_{\mathcal{T}})(\mathfrak{j}) = \begin{cases} \sup_{\mathfrak{h}\in\mathfrak{f}^{\rightarrow}(\mathfrak{j})} \eth_{\mathcal{T}}(\mathfrak{h}), & \textit{if } \mathfrak{f}^{\rightarrow}(\mathfrak{j}) \neq \textit{ptyset} \\ 0, & \textit{otherwise} \end{cases}$$

$$(1 - \mathfrak{f}(1 - \Upsilon_{\mathcal{T}}))(\mathfrak{j}) = \begin{cases} inf_{\mathfrak{h}\in\mathfrak{f}^{\rightarrow}(\mathfrak{j})} \Upsilon_{\mathcal{T}}(\mathfrak{h}), & \textit{if and only if } \mathfrak{f}^{\rightarrow}(\mathfrak{j}) \neq \textit{ptyset} \\ \Omega, & \textit{otherwise} \end{cases}$$

*Let us introduce the* symbol $\mathfrak{f} - (\Upsilon_{\mathcal{T}})$ *for* $(1 - \mathfrak{f}(1 - \Upsilon_{\mathcal{T}}))$

**Corollary 3.1.** *Let $\mathcal{M}_{\mathfrak{n}}$ be a neutrosophic over soft set in $\mathcal{M}(\forall \mathfrak{n} = 1, 2, \dots)$, $\mathcal{W}_{\mathfrak{m}}$ be a neutrosophic over soft set in $\mathcal{T}(\forall \mathfrak{m} = 1, 2, \dots)$ and $\mathfrak{f} : \mathcal{M} \to \mathcal{T}$ be a function. Then*

(i) $\mathcal{T}_1 \subseteq \mathcal{T}_2 \implies \mathfrak{f}(\mathcal{T}_1) \subseteq \mathfrak{f}(\mathcal{T}_2)$
(ii) $\mathcal{W}_1 \subseteq \mathcal{W}_2 \implies \mathfrak{f}^{\rightarrow}(\mathcal{W}_1) \subseteq \mathfrak{f}^{\rightarrow}(\mathcal{W}_2)$
(iii) $\mathcal{T} \subseteq \mathfrak{f}^{\rightarrow}(\mathfrak{f}(\mathcal{T}))\{$*if* $\mathfrak{f}$ *is injective, then* $\mathcal{T} = \mathfrak{f}^{\rightarrow}(\mathfrak{f}(\mathcal{T}))\}$
(iv) $\mathfrak{f}(\mathfrak{f}^{\rightarrow}(\mathcal{W})) \subseteq \mathcal{W}\{$*if* $\mathfrak{f}$ *is surjective, then* $\mathfrak{f}(\mathfrak{f}^{\rightarrow}(\mathcal{W})) = \mathcal{W}\}$
(v) $\mathfrak{f}^{\rightarrow}(\mho\mathcal{W}_{\mathfrak{m}}) = \mho\mathfrak{f}^{\rightarrow}(\mathcal{W}_{\mathfrak{m}})$
(vi) $\mathfrak{f}^{\rightarrow}(\Omega\mathcal{W}_{\mathfrak{m}}) = \Omega\mathfrak{f}^{\rightarrow}(\mathcal{W}_{\mathfrak{m}})$
(viii) $\mathfrak{f}(\mho\mathcal{T}_{\mathfrak{n}}) = \mho\mathfrak{f}(\mathcal{T}_{\mathfrak{n}})$
(viii) $\mathfrak{f}(\Omega\mathcal{T}_{\mathfrak{n}}) \subseteq \Omega\mathfrak{f}(\mathcal{T}_{\mathfrak{n}})\{$*if* $\mathfrak{f}$ *is injective, then* $\mathfrak{f}(\Omega\mathcal{T}_{\mathfrak{n}}) = \Omega\mathfrak{f}(\mathcal{T}_{\mathfrak{n}})\}$
(ix) $\mathfrak{f}^{\rightarrow}(\triangleright) = \triangleright$
(x) $\mathfrak{f}^{\rightarrow}(\blacktriangleright) = \blacktriangleright$
(xi) $\mathfrak{f}(\triangleright) = \triangleright$, *if* $\mathfrak{f}$ *is surjective*
(xii) $\mathfrak{f}(\blacktriangleright) = \blacktriangleright$
(xiii) $(\mathfrak{f}(\mathcal{T}))^{\complement} \subseteq \mathfrak{f}(\mathcal{T}^{\complement})$, *if* $\mathfrak{f}$ *is surjective*
(xiv) $\mathfrak{f}(\mathcal{T}^{\complement}) = (\mathfrak{f}(\mathcal{T}))^{\complement}$

## 4. Neutrosophic Over Soft Modal $\alpha^{\mathfrak{m}}$ Continuous Function

**Definition 4.1.** *Let $(\mathcal{M}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space. Then for a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\mathfrak{T}$ is called $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ closed set if $\circledast(\odot(\circledast(\mathfrak{T}))) \subseteq \mathfrak{T}$.*

**Definition 4.2.** *Let $(\mathcal{M}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space. Then for a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set $\mathfrak{T}$ is called $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set if $\odot(\circledast(\mathfrak{T})) \subseteq \mathcal{U}$ whenever $\mathfrak{T} \subseteq \mathcal{U}$ and $\mathcal{U}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ open set.*

**Remark 4.1.** *In a $(\mathcal{M}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ the following statements are true:*

(i) *Every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-closed set is both an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$-closed set and an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$-closed set.*
(ii) *Every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ closed set.*
(iii) *Every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ closed set is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set.*

**Proposition 4.1.** *If $\mathfrak{T}$ and $\mathcal{W}$ are two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed sets in $(\mathcal{M}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ then $\mathfrak{T}\mho\mathcal{W}$ is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set in $\mathcal{M}$.*

*Proof.* Let $\mathcal{U}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ open set in $(\mathcal{M}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ such that, $\mathfrak{T}\mho\mathcal{W} \subseteq \mathcal{U}$.
Now, $\odot(\circledast(\mathfrak{T}\mho\mathcal{W})) \subseteq \odot(\circledast(\mathfrak{T}))\mho\odot(\circledast(\mathcal{W})) \subseteq \mathcal{U}$.
Hence $\mathfrak{T}\mho\mathcal{W}$ is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set in $\mathcal{M}$. □

**Proposition 4.2.** *The intersection of an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set and an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set*

*Proof.* Let $\mathfrak{T}$ be an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set and $\mathcal{W}$ be an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set.
Since $\mathfrak{T}$ is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set.
$\odot(\circledast(\mathfrak{T})) \subseteq \mathcal{U}$ Whenever $\mathfrak{T} \subseteq \mathcal{U}$

where $\mathcal{U}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ open set.

To show that $\mathfrak{T}\Omega\mathcal{W}$ is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set it is enough to show that

$$\odot(\circledast(\mathfrak{T}\Omega\mathcal{W})) \subseteq \mathcal{U} \text{ Whenever } \mathfrak{T}\Omega\mathcal{W} \subseteq \mathcal{U}$$

where $\mathcal{U}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ open set.

Let $\mathcal{V} = \mathcal{W}^{\complement}$. Then $\mathfrak{T} \subseteq \mathcal{U}\Omega\mathcal{V}$

Since $\mathcal{V}$ is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ open set, $\mathcal{U}\Omega\mathcal{V}$ is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ open set and $\mathfrak{T}$ is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set,

$$\odot(\circledast(\mathfrak{T})) \subseteq \mathcal{U}\Omega\mathcal{V}$$

$$\begin{aligned}\odot(\circledast(\mathfrak{T}\Omega\mathcal{W})) &\subseteq \odot(\circledast(\mathfrak{T}))\Omega\odot(\circledast(\mathcal{W}))\\ &\subseteq \odot(\circledast(\mathfrak{T}))\Omega\mathcal{W} \subseteq (\mathcal{U}\Omega\mathcal{V})\Omega\mathcal{W}\\ &\subseteq (\mathcal{U}\Omega\mathcal{W})\Omega(\mathcal{V}\Omega\mathcal{W})\\ &\subseteq (\mathcal{U}\Omega\mathcal{W})\Omega \rhd \subseteq \mathfrak{T}\Omega\mathcal{W}\end{aligned}$$

$\Longrightarrow$ $\mathfrak{T}\Omega\mathcal{W}$ is an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set. □

**Proposition 4.3.** *Every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set if and only if $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ closed set.*

*Proof.* The proof of this proposition is straightforward and self-evident. □

**Definition 4.3.** *Let $(\mathcal{M}, \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space and $\mathfrak{T}$ be an $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set .Then the $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ interior and $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closure of $\mathfrak{T}$ is defined by*

(i) $\odot_{\alpha^{\mathfrak{m}}}(\mathfrak{T}) = \mho\{\mathcal{N} : \mathcal{N} \subseteq \mathfrak{T} \quad and \quad \mathcal{N} \in \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}\}$.

(ii) $\circledast_{\alpha^{\mathfrak{m}}}(\mathfrak{T}) = \Omega\{\mathcal{O} : \mathfrak{T} \subseteq \mathcal{O} \quad and \quad \mathcal{O}^{\complement} \in \tau_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}\}$.

**Definition 4.4.** *Let $(\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ and $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be any two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological spaces. Then a map $\mathfrak{f} : (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is called*

(i) *$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous if the inverse image of every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set in $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set in $(\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$.*

(ii) *$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ continuous if the inverse image of every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set in $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ closed set in $(\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$.*

(iii) *$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous if the inverse image of every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set in $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set in $(\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$.*

**Proposition 4.4.** *Let $(\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ and $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be any two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological spaces. If $\mathfrak{f} : (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous, then it is*

(i) *$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ continuous.*

(ii) *$\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous.*

*Proof.* Assume that $\mathfrak{f} : (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous. Then,

$\mathfrak{T}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-closed set in $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$.

(i) Since $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous,

$$\mathfrak{f}^{-1}(\mathfrak{T}) \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ closed set in } (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Now, every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set is also a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ closed set. Hence:

$$\mathfrak{f}^{-1}(\mathfrak{T}) \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ modal } \alpha \text{ closed set in } \mathcal{M}_1.$$

Therefore, $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ continuous.

(ii) Since $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous,

$$\mathfrak{f}^{-1}(\mathfrak{T}) \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ closed set in } (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Now, every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set is also a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set. Hence:

$$\mathfrak{f}^{-1}(\mathfrak{T}) \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ modal } \alpha^{\mathfrak{m}} \text{ closed set in } \mathcal{M}_1.$$

Therefore, $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous. □

**Remark 4.2.** *Converse of proportion 4.4 is may not be true. Its proved in example 4.1.*

**Example 4.1.** *Let $\mathcal{M}_1 = \{\mathfrak{h}\}$, $\mathcal{M}_2 = \{\mathfrak{h}\}$ and $\Omega = 1.5$ be two universes. Define $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topologies on both as follows: Let*

$$\tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}} = \left\{\triangleright, \blacktriangleright, \mathfrak{T}^{\mathfrak{C}}\right\},$$

*where*

$$\mathfrak{T}^{\mathfrak{C}} = \{\langle \mathfrak{h}, 0.7, 0.7, 1.3\rangle\},$$

*Let*

$$\tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}} = \{\triangleright, \blacktriangleright, \mathcal{W}\},$$

*where*

$$\mathcal{W} = \{\langle \mathfrak{h}, 1.2, 0.9, 0.6\rangle\},$$

*Define $\mathfrak{f} : (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ by*

$$\mathfrak{f}(h) = h$$

*(i) Let $\mathfrak{T}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ closed set in $(\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$. Then*

$$\mathfrak{f}(\mathfrak{T}) = \mathfrak{T}.$$

*Here $\mathfrak{T}$ is not $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set in $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$*
*Therefore, $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ continuous, but it is not $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous.*

*(ii) Let $\mathfrak{T}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set in $(\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$. Then*

$$\mathfrak{f}(\mathfrak{T}) = \mathfrak{T}.$$

*Here $\mathfrak{T}$ is not $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set in $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$*
*Therefore, $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous, but it is not $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous.*

**Proposition 4.5.** *Let $(\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ and $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be any two $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological spaces. If $\mathfrak{f} : (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous if and only if $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ continuous.*

*Proof.* Assume that $\mathfrak{f} : (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous. Let $\mathfrak{T}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-closed set in $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$.

Then, by assumption:

$$\mathfrak{f}^{-1}(\mathfrak{T}) \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ modal } \alpha^{\mathfrak{m}} \text{ closed set in } (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Now, every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set is also a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ closed set. Hence:

$$\mathfrak{f}^{-1}(\mathfrak{T}) \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ modal } \alpha \text{ closed set in } (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Therefore, $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ continuous.

Conversely, Assume that $\mathfrak{f} : (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ continuous. Let $\mathfrak{T}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-closed set in $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$.

Then, by assumption:

$$\mathfrak{f}^{-1}(\mathfrak{T}) \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ modal } \alpha \text{ closed set in } (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Now, every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ closed set is also a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ closed set. Hence:

$$\mathfrak{f}^{-1}(\mathfrak{T}) \text{ is a } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ modal } \alpha^{\mathfrak{m}} \text{ closed set in } (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Therefore, $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha$ continuous. □

**Proposition 4.6.** *Let $(\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$, $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$, and $(\mathcal{M}_3, \tau^3_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological spaces. Let $\mathfrak{f} : (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ and $\mathfrak{g} : (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_3, \tau^3_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be functions. Then:*

(i) *If both* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *continuous, then* $\mathfrak{g} \circ \mathfrak{f}$ *is* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *continuous.*
(ii) *If both* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *modal continuous, then* $\mathfrak{g} \circ \mathfrak{f}$ *is* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *modal continuous.*
(iii) *If both* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *modal* $\alpha^{\mathfrak{m}}$ *continuous, then* $\mathfrak{g} \circ \mathfrak{f}$ *is* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ *modal* $\alpha^{\mathfrak{m}}$ *continuous.*

*Proof.* Let $\mathfrak{f} : (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ and $\mathfrak{g} : (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}) \to (\mathcal{M}_3, \tau^3_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$ be functions.

(i) Suppose both $\mathfrak{f}$ and $\mathfrak{g}$ are $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous.
Let $\mathfrak{T}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set in $(\mathcal{M}_3, \tau^3_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$. Since $\mathfrak{g}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous, we have:

$$\mathfrak{g}^{-1}(\mathfrak{T}) \text{ is } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ closed in } (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Again, since $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous, it follows that:

$$\mathfrak{f}^{-1}(\mathfrak{g}^{-1}(\mathfrak{T})) = (\mathfrak{g} \circ \mathfrak{f})^{-1}(\mathfrak{T}) \text{ is } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ closed in } (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Hence, $\mathfrak{g} \circ \mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ continuous.

(ii) Suppose both $\mathfrak{f}$ and $\mathfrak{g}$ are $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal continuous.
Let $\mathfrak{T}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set in $(\mathcal{M}_3, \tau^3_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$.
Then, $\mathfrak{g}^{-1}(\mathfrak{T})$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal closed in $(\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$, and hence:

$$\mathfrak{f}^{-1}(\mathfrak{g}^{-1}(\mathfrak{T})) = (\mathfrak{g} \circ \mathfrak{f})^{-1}(\mathfrak{T}) \text{ is } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ modal closed in } (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Therefore, $\mathfrak{g} \circ \mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal continuous.

(iii) Suppose both $\mathfrak{f}$ and $\mathfrak{g}$ are $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous.
Let $\mathfrak{T} \subseteq \rho(\mathcal{M}_3)$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ closed set in $(\mathcal{M}_3, \tau^3_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}})$.
Since $\mathfrak{g}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous, we have:

$$\mathfrak{g}^{-1}(\mathfrak{T}) \text{ is } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ modal } \alpha^{\mathfrak{m}} \text{ closed in } (\mathcal{M}_2, \tau^2_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Then since $\mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous, it follows that:

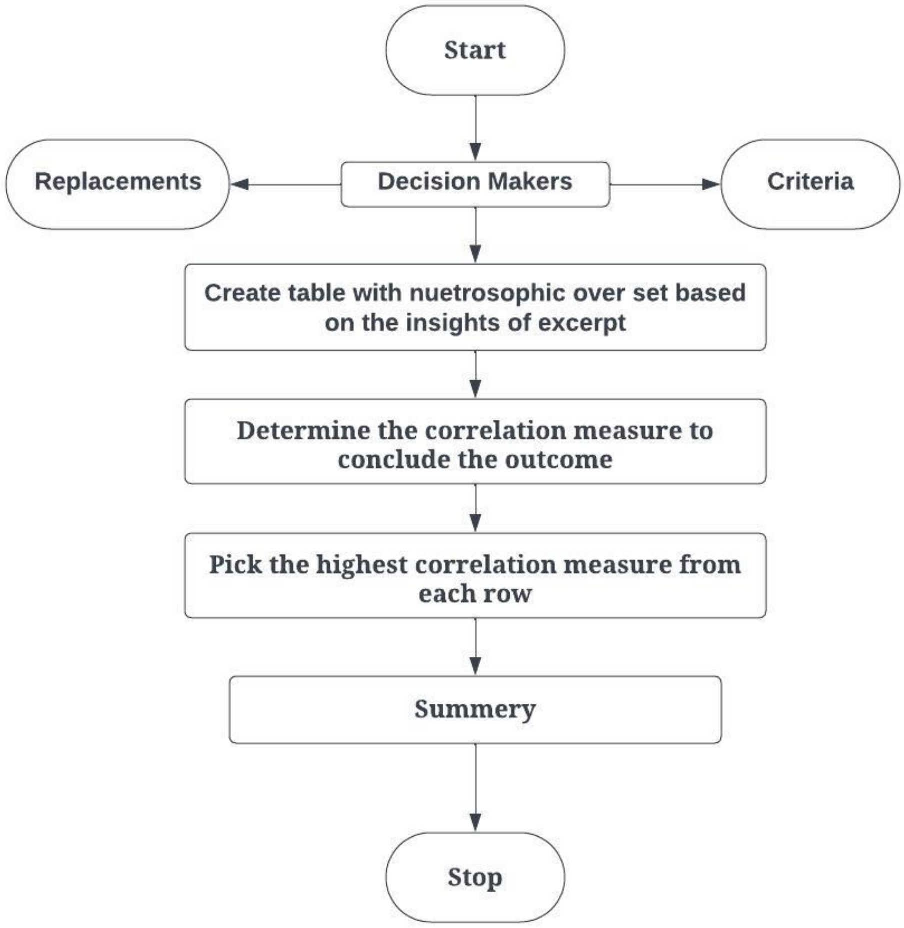
$$(\mathfrak{g} \circ \mathfrak{f})^{-1}(\mathfrak{T}) = \mathfrak{f}^{-1}(\mathfrak{g}^{-1}(\mathfrak{T})) \text{ is } \mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}} \text{ modal } \alpha^{\mathfrak{m}} \text{ closed in } (\mathcal{M}_1, \tau^1_{\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}}).$$

Hence, $\mathfrak{g} \circ \mathfrak{f}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ modal $\alpha^{\mathfrak{m}}$ continuous. □

**Definition 4.5.** *Let* $\mathcal{T}_i$ *and* $\mathcal{W}i$ *be any two* $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$*-sets and* $\mathfrak{n}$ *be a criteria for the problem. Then cosine similarity measure(*$\wp$*) is calculated by*

$$\wp = \frac{1}{\mathfrak{n}} \sum_{i=1}^{\mathfrak{n}} \left\{ \frac{\blacktriangleright_{\mathcal{T}_i}(\mathfrak{h}) \blacktriangleright_{\mathcal{W}_i}(\mathfrak{h}) + \eth_{\mathcal{T}_i}(\mathfrak{h}) \eth_{\mathcal{W}_i}(\mathfrak{h}) + \blacktriangleleft_{\mathcal{T}_i}(\mathfrak{h}) \blacktriangleleft_{\mathcal{W}_i}(\mathfrak{h})}{\sqrt{\blacktriangleright^2_{\mathcal{T}_i}(\mathfrak{h}) + \eth^2_{\mathcal{T}_i}(\mathfrak{h}) + \blacktriangleleft^2_{\mathcal{T}_i}(\mathfrak{h})} \sqrt{\blacktriangleright^2_{\mathcal{W}_i}(\mathfrak{h}) + \eth^2_{\mathcal{W}_i}(\mathfrak{h}) + \blacktriangleleft^2_{\mathcal{W}_i}(\mathfrak{h})}} \right\}$$

## 5. Flowchart To Calculate Similarity Measure of Correlation for $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set




## 6. Numerical Application

To this day, one of the primary threats to human health remains cancer, and this is the reason the diseases listed under the term 'cancer' are eternally not tiring from attempts to understand better and prevent or treat them. Advancement in detection procedures such as improved screening campaigns could assist tremendously to identify when cancer is still manageable. These medical procedures are broadly categorized as surgery, chemotherapy, radiation therapy, immunotherapy, and also targeted therapy, and are continually evolving. Immunotherapy, using the patient's immune system for cancer treatment, will probably work for many different kinds of cancer. The treatment methodologies of the present era are becoming more individualized and feature a basis on the success of the genetic make-up of the patient.

Staying strong in relentless efforts culminates into new discoveries about the underlying molecular mechanism of cancer, building novel therapeutic avenues and ideas. Such remarkable milestones, however, haven't signified the cessation of decades and decades of cancer preventive strategies like dietary and vaccination changes over the years. Therefore, even though very significant progress has been made, there shall be ever more need for cooperation among scientists, clinicians, and pharmaceutical companies to completely eradicate this disease from the world. There are now several hospitals that provide treatment for cancer patients, but it is difficult to find a good hospital.

A good hospital that can treat some people's cancers is the primary goal of every cancer victim. But judging such hospitals becomes a very complicated decision-making process. It permits patients to make a comparison of treatment success rates, quality of care, types of facilities available, and costs at different hospitals, and therefore come closer to making the ultimate decision regarding which hospital serves them best. For the patient, their families, and other stakeholders in choice alternatives where the first choice is sometimes not available or appropriate, these parameters come to the decision-making adding value. All this data creates a holistic view under which a patient can find the best solutions meeting his and her medical needs and wants for a long time and the best yields in health outcome availability.

Let $\mathfrak{B}_1, \mathfrak{B}_2$ and $\mathfrak{B}_3$ be the three hospitals chosen by the patient. The decision-makers must select the best among these three hospitals. The criterion comprises Low cost ($\mathfrak{C}_1$), Most recoveries ($\mathfrak{C}_2$), and Infrastructure ($\mathfrak{C}_3$). As for the alternatives, they would be $\mathfrak{D}_1$, $\mathfrak{D}_2$, and $\mathfrak{D}_3$, for the first, second, and third places respectively.

Table 1 presents the relationship between various hospitals and the required selection criteria, while Table 2 illustrates how these criteria align with the available replacements. Table 3 introduces the cosine similarity measure used for comparative analysis. Figure 1 provides a graphical view of the hospital selection process based on these measures. Further detailed ranking outcomes are shown in Table 4, with Figure 2 offering a corresponding visual representation. Finally, Table 5 gives a summarized overview of the results, supporting effective decision-making.

Table 1. Relation between Hospitals and Required criteria

| $\mathcal{T}$ | $\mathfrak{C}_1$ | $\mathfrak{C}_2$ | $\mathfrak{C}_3$ |
|---|---|---|---|
| $\mathfrak{B}_1$ | (1.2,0.5,0.3) | (1.02,0.6,0.3) | (0.6,0.9,1.04) |
| $\mathfrak{B}_2$ | (1.2,0.7,0.5) | (1.06,0.4,0.6) | (1.24,0.56,0.1) |
| $\mathfrak{B}_3$ | (1.03,1.01,0.05) | (0.58,1.25,0.36) | (1.04,0.82,0.12) |

Table 2. Relation between Required criteria and replacement

| $\mathcal{W}$ | $\mathfrak{D}_1$ | $\mathfrak{D}_2$ | $\mathfrak{D}_3$ |
|---|---|---|---|
| $\mathfrak{C}_1$ | (1.15,0.3,0.42) | (1.18,0.7,0.36) | (1.36,0.19,0.24) |
| $\mathfrak{C}_1$ | (1.06,0.07,0.6) | (1.12,0.17,0.69) | (1.42,0.16,0.33) |
| $\mathfrak{C}_1$ | (0.18,1.5,0.86) | (1.33,1.1,0.1) | (1.04,0.48,0.35) |

Table 3. Cosine Similarity Measure

| $\wp$ | $\mathfrak{D}_1$ | $\mathfrak{D}_2$ | $\mathfrak{D}_3$ |
|---|---|---|---|
| $\mathfrak{B}_1$ | **0.9204** | 0.8719 | 0.8917 |
| $\mathfrak{B}_2$ | 0.8060 | **0.9807** | 0.9436 |
| $\mathfrak{B}_3$ | 0.6717 | **0.8449** | 0.7656 |

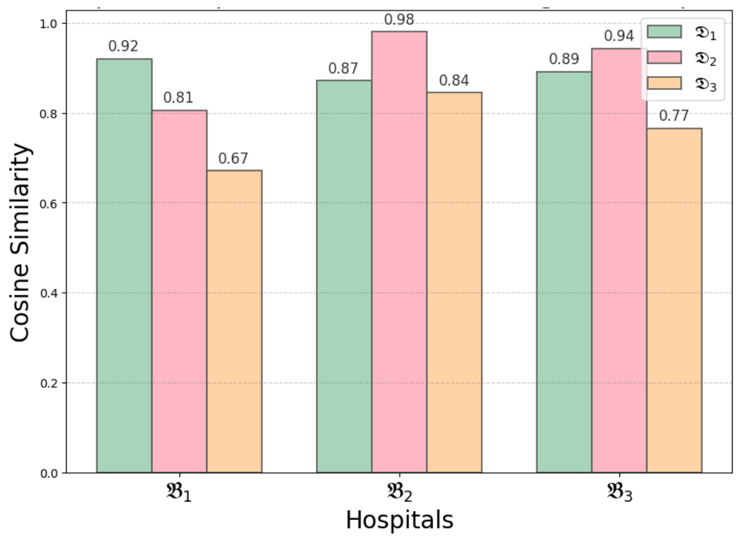



FIGURE 1. Graphical Representation for Selecting Best Hospitals

TABLE 4. Cosine Similarity Measure for First Ranking

| $\wp$ | $\mathfrak{D}_1$ |
|---|---|
| $\mathfrak{B}_1$ | **0.9204** |
| $\mathfrak{B}_2$ | 0.8060 |
| $\mathfrak{B}_3$ | 0.6717 |

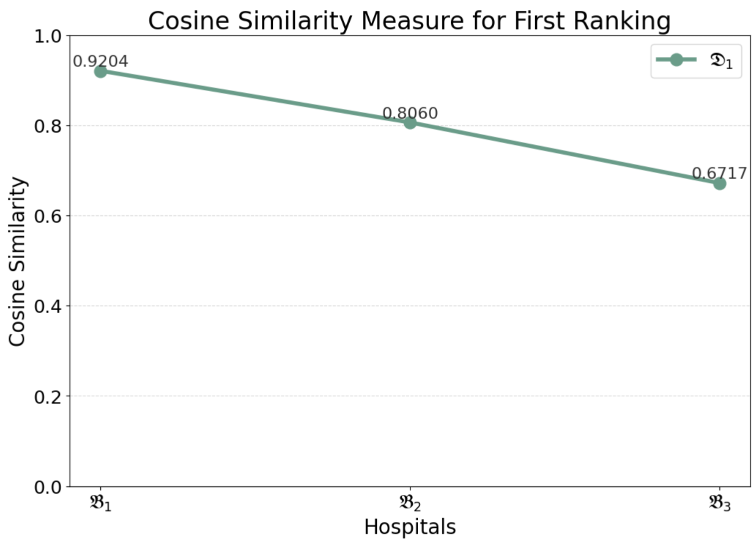



FIGURE 2. Graphical Representation for Cosine Similarity Measure for First Ranking

Table 5. Summary

| $\wp$ | **Place** |
|---|---|
| $\mathfrak{B}_1$ | Selected |
| $\mathfrak{B}_2$ | Non-Selected |
| $\mathfrak{B}_3$ | Non-Selected |

## 7. Conclusions

There is new methodology introduced within neutrosophic over soft modal topological structure, which gives birth to a neutrosophic over soft modal topological continuous function. This will become an important tool when studying continuity in light of the neutrosophic over-model, that is uncertain and vague. To further display one of the applications for neutrosophic over-soft set, the cosine similarity measuring technique is adopted-giving it a numeric example of application and almost an open door for other applications that one would like to set for finer comparisons. A few very important properties and operations are further subjected to discussion, in that respect influencing the mode of paying attention to an alternate option. Thus, this paper applies the model to rank hospitals using a case study, with $\mathfrak{B}_1$ ranked first, $\mathfrak{B}_2$ second, and $\mathfrak{B}_3$ third. Therefore, $\mathfrak{B}_1$ is the recommended best option for cancer treatment, proving that the modal works perfectly and can apply in real-life problem choice in healthcare by continuous function.

Similarly, all classification contingency and interaction measures can travel across different domains of application, from medicine through industry and construction, to utilizing the $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-set correlation measure.

## References

[1] Atanassov, K. T., On intuitionistic fuzzy sets theory, Springer.

[2] Atanassov, K., (2022), Intuitionistic fuzzy modal topological structure, Mathematics, 10(18), p. 3313.

[3] Bellman, R. E. and Zadeh, L. A., (1970), Decision-making in a fuzzy environment, Management Science.

[4] Broumi, S., (2013), Generalized neutrosophic soft set, International Journal of Computer Science, Engineering and Information Technology, 3(2), pp. 17-30.

[5] Bustince, H. and Burillo, P., (1995), Correlation of interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, 74(2), pp. 237-244.

[6] Christianto, V., Boyd, R. N. and Smarandache, F., (2020), Three possible applications of neutrosophic logic in fundamental and applied sciences, International Journal of Neutrosophic Science, 1(2), pp. 90-95.

[7] Devi, R. N. and Parthiban, Y., (2024), Decision making by neutrosophic over soft topological space, Neutrosophic Sets and Systems, 66, pp. 76-94.

[8] Devi, R. N. and Parthiban, Y., (2024), Enhancing decision-making for parents: A neutrosophic Pythagorean plithogenic hypersoft set approach to school selection with TOPSIS method, Multi-Criteria Decision Making Models and Techniques: Neutrosophic Approaches, IGI Global, pp. 1-20.

[9] Devi, R. N. and Parthiban, Y., (2024), Exploring neutrosophic over supra exterior modal topological structure: Theory and application in healthcare decision-making, Neutrosophic Sets and Systems, 72, pp. 272-285.

[10] Devi, R. N. and Parthiban, Y., (2024), Analyzing pharmaceutical compounds through neutrosophic over soft hyperconnected spaces, Orient J Chem, 40(6), pp. 1627-1633.

[11] Dhavaseelan, R., Devi, R. N., Jafari, S. and Imran, Q. H., (2019), Neutrosophic alpha m-continuity, Neutrosophic Sets and Systems, 27, pp. 171-179.

[12] Jansi, R. , Mohana, K. and Smarandache, F., (2019), Correlation measure for Pythagorean neutrosophic sets with T and F as dependent neutrosophic components, Neutrosophic Sets and Systems, 30, pp. 202-212.

[13] Kumaravel, S. K., Murugesan, R., Nagaram, N. B., Rasappan, S. and Yamini, G., (2023), Applications of fuzzy cognitive maps and neutrosophic cognitive maps on analysis of dengue fever, Fuzzy Logic Applications in Computer Science and Mathematics, pp. 249-265.

[14] Maji, P. K., Biswas, R. and Roy, A. R., (2003), Soft set theory, Computers & Mathematics with Applications, 45(4-5), pp. 555-562.

[15] Mallick, R. and Pramanik, S., (2020), Pentapartitioned neutrosophic set and its properties, Neutrosophic Sets and Systems, 36, 184-192.

[16] Molodtsov, D., (1999), Soft set theory—first results, Computers & Mathematics with Applications, 37(4-5), pp. 19-31.

[17] Murugesan, R., Parthiban, Y., Devi, R., Mohan, K. R. and Kumaravel, S. K., (2023), A comparative study of fuzzy cognitive maps and neutrosophic cognitive maps on COVID variants, Neutrosophic Sets and Systems, 55(1), p. 20.

[18] Radha, R., Mary, A. S. A., Prema, R. and Broumi, S., (2021), Neutrosophic Pythagorean sets with dependent neutrosophic Pythagorean components and its improved correlation coefficients, Neutrosophic Sets and Systems, 46, pp. 77-86.

[19] Saeed, M. , Saqlain, M., Mehmood, A. and Yaqoob, S., (2020), Multi-polar neutrosophic soft sets with application in medical diagnosis and decision-making, Neutrosophic Sets and Systems, 33, pp. 183-207.

[20] Smarandache, F., (1999), A unifying field in logics: Neutrosophic logic, American Research Press, pp. 1-141.

[21] Smarandache, F., (2016), Neutrosophic overset, neutrosophic underset, and neutrosophic offset, Pons Editions Brussels.

[22] Wang, H., Smarandache, F., Zhang, Y. and Sunderraman, R., (2010), Single valued neutrosophic sets, North-European Scientific Publishers, 12, pp. 410-413,.

[23] Ye, J., (2013), Another form of correlation coefficient between single valued neutrosophic sets and its multiple attribute decision-making method, Neutrosophic Sets and Systems, 1(1), pp. 8-12.

[24] Zadeh, L. A., (1965), Fuzzy sets, Information and Control, 8(3), pp. 338-353.

[25] Zadeh, L. A., (1978), Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems, 1(1), pp. 3-28.

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