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# DISTANCE $\tau$-POLYNOMIAL AND EDGE DISTANCE $\tau$-POLYNOMIAL OF GRAPHS

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ABSTRACT. The distance $\tau$-polynomial $\mathbb{D}(G,x)$ of a graph $G=(V,E)$ is defined as $\mathbb{D}(G,x)=\sum_{k=0}^{\binom{n}{2}}\tau_k x^k$, where $\tau_k$ is the number of vertices with $\tau_G(u)=\sum_{x\in V(G)} d_G(u,x)=k, \forall u\in V(G)$. In this paper, we seek to find the distance $\tau$-polynomial and similarly, edge distance $\tau$-polynomial of some graph families and graph operations.

Keywords: closeness centrality, edge closeness centrality, distance $\tau$-polynomial, edge distance $\tau$-polynomial.

AMS Subject Classification: 05C12, 05C31, 05C99.

## 1. INTRODUCTION

In graph theory, a graph polynomial is an invariant associated with a graph that is expressed as a polynomial. These types of invariants are studied within the field of algebraic graph theory. Numerous polynomials are linked with graphs, such as domination polynomial, Hosoya polynomial, clique polynomial, characteristic polynomial, chromatic polynomial, matching polynomial and Tutte polynomial. These polynomials serve as valuable tools in both algebraic graph theory and combinatorics, offering various methods for analyzing and classifying graphs based on their structural features. Ongoing research explores their connections and applications in diverse areas, including optimization, chemistry, and network theory. For more information on graph polynomials, one can refer additional resources in the literature [2, 6, 10, 13, 19].

Let $G$ be a nontrivial, connected, finite undirected graph with vertex set $V(G)$ and edge set $E(G)$. Further, the number of vertices is order of $G$ and edges is size of $G$. If

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$u, v \in V(G)$ are adjacent then we denote it by $u \sim v$[5], the degree of $v$ is denoted by $d(v)$ or $d_G(v)$ and is defined as the number of edges incident with $v$, the distance between $u$ and $v$ is denoted by $d(u, v)$ or $d_G(u, v)$ and is defined as the length of the shortest path connecting $u$ and $v$ in $G$[12]. If $e(= uv) \in E(G)$, then $d(e)$ or $d_G(e)$ denotes the degree of $e$ and $d(e) = d(u) + d(v) - 2$[14]. The vertices of the line graph $L(G)$ are the edges of $G$ with two vertices of line graph adjacent whenever the corresponding edges of $G$ are adjacent[12]. The distance $d(e, f)$ or $d_G(e, f)$ between two edges $e$ and $f$ is defined as the distance between the corresponding vertices in the line graph of $G$. The complement of $G$ is a graph $\overline{G}$, with vertex set same as of $G$ and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in $G$ and $G$ is said to be self-complementary graph if $G$ is isomorphic to $\overline{G}$[12]. Complement of line graph is the jump graph $J(G)$ of $G$. The helm graph $H_n$ is the graph constructed by adjoining a pendant edge at each vertex of the $n$-cycle in a wheel graph $W_{n+1}$[21]. A gear graph $G_n$ is obtained from the wheel graph by adding a vertex between every pair of adjacent vertices of the $n$-cycle[22]. A flower graph $F_n$ is obtained from a helm graph by joining each pendant vertex to the central vertex by an edge[4]. The friendship graph $C_3^n$ is a collection of $n$ triangles with a common vertex [11]. A fan graph $F_{1,n}$, $n \geq 2$ is obtained by joining all vertices $u_1, u_2, u_3, ..., u_n$ of $P_n$ to a further vertex $u_{n+1}$ called center[18]. The $Pl_n$ graph ($n \geq 3$) is a graph obtained by the join of $P_{n-2}$ and $P_2$[20]. A barbell graph $B_n$ is the graph obtained by adding an edge between two copies of $K_n, n \geq 3$[1]. A complete split graph $CS(n, k)$ of order $n + k$ with $V(CS(n, k)) = A \cup B$, where $A = \{a_1, a_2, ..., a_n\}$ induces a clique and $B = \{b_1, b_2, ..., b_k\}$ induces a null graph such that every $a_i b_j$ is an edge, $1 \leq i \leq n, 1 \leq j \leq k$[9]. The shadow graph $S(G)$ of $G$ is the graph obtained by taking two copies of $G$, say $G_1$ and $G_2$, and then joining each vertex $v \in V(G_1)$ to the neighbors of $v' \in V(G_2)$, where $v'$ is the vertex in $V(G_2)$ corresponding to $v$[3]. The total graph $T(G)$ has a vertex set $V(G) \cup E(G)$, and two vertices of $T(G)$ are adjacent whenever they are neighbors in $G$ [12]. The join of $G_1$ and $G_2$ is the graph $G = G_1 + G_2$ having the vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$ [12]. The cartesian product of $G_1$ and $G_2$ is $G = G_1 \times G_2$, with the vertex set $V(G) = V(G_1) \times V(G_2)$, where two distinct vertices $(u, v)$ and $(x, y)$ of $G_1 \times G_2$ are adjacent if either $u = x$ and $vy \in E(G_2)$ or $v = y$ and $ux \in E(G_1)$ [12]. To determin e the value of $\tau_G(u)$ one can refer the following definitions and theorems.

**Definition 1.1.** [9] *Let $G$ be a nontrivial connected graph of order $n$ and $u \in V(G)$, then the closeness centrality of $u$ is given by* $\mathcal{C}_G(u) = \dfrac{n-1}{\tau_G(u)}$, *where* $\tau_G(u) = \sum\limits_{x \in V(G)} d_G(u, x)$.

**Definition 1.2.** [16] *For a connected graph $G = (V, E)$ with $m \geq 2$, the edge closeness centrality of an edge $e$ is given by* $E\mathcal{C}_G(e) = \dfrac{m-1}{\tau_G(e)}$, *where* $\tau_G(e) = \sum\limits_{f \in E(G)} d_G(e, f)$.

**Theorem 1.1.** [8] *Let $G_1 + G_2$ be the join of two graphs $G_1$ and $G_2$, where $V(G_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ with both $m, n \geq 1$. Then the closeness centrality of vertices $u_i$ and $v_j$ in $G_1 + G_2$ are given by the following expressions:*

$$\mathcal{C}_G(u_i) = \frac{m+n-1}{2m+n-2-deg_{G_1}(u_i)} \text{ for } 1 \leq i \leq m.$$

$$\mathcal{C}_G(v_j) = \frac{m+n-1}{2n+m-2-deg_{G_2}(v_j)} \text{ for } 1 \leq j \leq n.$$

**Theorem 1.2.** [3] *Let $G$ be a nontrivial connected graph. For each $p \in V(G_1)$ and $p' \in V(G_2)$, $\tau_{S(G)}(p) = 2(\tau_{G_1}(p)) + 2$ and $\tau_{S(G)}(p') = 2(\tau_{G_2}(p')) + 2$.*

**Theorem 1.3.** [17] *Let $G$ be a nontrivial graph of order $p$ and $v \in V(G)$. Then* $\tau_G(v) \leq \dfrac{p(p-1)}{2} = \binom{p}{2}$.

## 2. Distance $\tau$-polynomial

**Definition 2.1.** *The distance $\tau$-polynomial $\mathbb{D}(G,x)$ of a graph $G$ is defined as*

$$\mathbb{D}(G,x) = \sum_{k=0}^{\binom{n}{2}} \tau_k x^k,$$

*where $\tau_k$ is the number of vertices with* $\tau_G(u) = \sum_{x \in V(G)} d_G(u,x) = k, \forall u \in V(G)$.

In the following proposition we provide the distance $\tau$-polynomial of some standard graphs.

**Proposition 2.1.** (1) *For complete graph $K_n$,*

$$\mathbb{D}(K_n,x) = nx^{n-1}.$$

(2) *For cycle graph $C_n$,*

$$\mathbb{D}(C_n,x) = \begin{cases} nx^{\frac{n^2-1}{4}}, & \text{if } n \text{ is odd;} \\ nx^{\frac{n^2}{4}}, & \text{if } n \text{ is even.} \end{cases}$$

(3) *For path graph $P_n$, $n \geq 2$,*

$$\mathbb{D}(P_n,x) = 2x^{\binom{n}{2}} + \sum_{i=2}^{n-1} x^{\binom{i}{2}+\binom{n+1-i}{2}}.$$

(4) *For wheel graph $W_{n+1}$, $n \geq 5$,*

$$\mathbb{D}(W_{n+1},x) = x^n + nx^{2n-3}.$$

(5) *For complete bipartite graph $K_{m,n}$, $m,n \geq 4$,*

$$\mathbb{D}(K_{m,n},x) = mx^{2m+n-2} + nx^{2n+m-2}.$$

*Proof.* The proof follows from Definition 2.1 and Table 1.

| Graph $G$ | $\tau_G(u), u \in V(G)$ | $\tau_k$ |
|---|---|---|
| Complete graph $(K_n)$ | $n-1$ | $n$ |
| Cycle $(C_n)$ | $\frac{n^2-1}{4}$, if $n$ is odd | $n$ |
| | $\frac{n^2}{4}$, if $n$ is even | $n$ |
| Path $(P_n), n \geq 2$ | $\binom{n}{2}$, if $u_i$ is a pendant vertex | 2 |
| | $\binom{i}{2} + \binom{n+1-i}{2}$, if $u_i$ is a nonpendant vertex | $n-2$ |
| Wheel $(W_{n+1}), n \geq 5$ | $n$, if $u$ is a universal vertex | 1 |
| | $2n-3$, if $u$ is not a universal vertex | $n$ |
| Complete bipartite $(K_{m,n}), m, n \geq 4$ | $2m+n-2$, if $deg(u) = n$ | $m$ |
| | $2n+m-2$, if $deg(u) = m$ | $n$ |

Table 1

One can refer [9] for the $\tau_G(u)$.

□

In the following proposition we provide the distance $\tau$-polynomial of some special graphs.

**Proposition 2.2.** (1) *For fan graph $F_{1,n}$, $n \geq 3$,*
$$\mathbb{D}(F_{1,n}, x) = x^n + 2x^{2n-2} + (n-2)x^{2n-3}.$$
(2) *For complete split graph $CS(n, k)$,*
$$\mathbb{D}(CS(n, k)), x) = nx^{n+k-1} + kx^{n+2(k-1)}.$$
(3) *For helm graph $H_n$,*
$$\mathbb{D}(H_n, x) = nx^{7n-8} + nx^{5n-7} + x^{3n}.$$
(4) *For gear graph $G_n$,*
$$\mathbb{D}(G_n, x) = nx^{7n-10} + nx^{5n-5} + x^{3n}.$$
(5) *For flower graph $F_n$,*
$$\mathbb{D}(F_n, x) = nx^{4n-4} + nx^{4n-2} + x^{2n}.$$
(6) *For friendship graph $C_3^n$,*
$$\mathbb{D}(C_3^n, x) = 2nx^{4n-2} + x^{2n}.$$
(7) *For $Pl_n$ graph,*
$$\mathbb{D}(Pl_n, x) = 2x^{n-1} + 2x^{2n-5} + (n-4)x^{2n-6}.$$

*Proof.* The proof follows from Definition 2.1 and Table 2.

| Graph $G$ | $\tau_G(u), u \in V(G)$ | $\tau_k$ |
|---|---|---|
| Fan graph $(F_{1,n}), n \geq 3$ | $n$, if $i = n+1$ | 1 |
| | $2(n-1)$, if $i = 1$ or $i = n$; | 2 |
| | $2n-3$, if $i = 2, 3, 4, \ldots, n-1, n \geq 3$ | $n-2$ |
| Complete split graph $(CS(n,k))$ | $n+k-1$, if $deg(u) = n+k-1$ | $n$ |
| | $n+2(k-1)$, if $deg(u) = n$ | $k$ |
| Helm graph $(H_n)$ | $7n-8$, if $u$ is a pendant vertex | $n$ |
| | $5n-7$, if $u$ is adjacent to central vertex | $n$ |
| | $3n$, if $u$ is a central vertex | 1 |
| Gear graph $(G_n)$ | $3n$, if $u$ is a central vertex | 1 |
| | $5n-5$, if $u$ is adjacent to central vertex | $n$ |
| | $7n-10$, if $u$ is not adjacent to central vertex | $n$ |
| Flower graph $(F_n)$ | $2n$, if $u$ is a universal vertex | 1 |
| | $4n-2$, if $deg(u) = 2$ | $n$ |
| | $4n-4$, if $deg(u) = 4$ | $n$ |
| Friendship graph $(C_3^n)$ | $2n$, if $u$ is a universal vertex | 1 |
| | $4n-2$, if $u$ is not a universal vertex | $2n$ |
| $Pl_n$ graph | $n-1$ , if $deg(u) = n-1$ | 2 |
| | $2n-5$, if $deg(u) = 3$ | 2 |
| | $2n-6$, if $deg(u) = 4$ | $n-4$ |

Table 2

One can refer [9, 16] for the $\tau_G(u)$.

□

**Observation 2.1.** (1) *The constant term of* $\mathbb{D}(G,x)$ *is always zero.*
(2) *Sum of all the coefficients in* $\mathbb{D}(G,x) = |V(G)|$.
(3) *The distance* $\tau$*-polynomial of the peterson graph* $P$ *is*

$$\mathbb{D}(P,x) = 10x^{15}.$$

## 3. Edge Distance $\tau$-polynomial

Motivated by the definition of distance $\tau$- polynomial of graphs. In this section, we define edge distance $\tau$-polynomial and computed for some standard graphs.

**Definition 3.1.** *The edge distance $\tau$-polynomial $\mathbb{ED}(G,x)$ of a graph $G$ defined as $\mathbb{ED}(G,x) = \sum \tau_j x^j$, where $\tau_j$ is the number of edges with $\tau_G(e) = \sum_{x\in E(G)} d_G(e,x) = j, \forall e \in E(G)$.*

In the following proposition we provide the edge distance $\tau$-polynomial of some standard graphs.

**Proposition 3.1.** (1) *For star graph $K_{1,n}$,*

$$\mathbb{ED}(K_{1,n},x) = nx^{n-1}.$$

(2) *For the path graph $P_n$, $n \geq 3$,*

$$\mathbb{ED}(P_n,x) = 2x^{\binom{n-1}{2}} + \sum_{i=2}^{n-2} x^{\binom{i}{2}+\binom{n-i}{2}}.$$

(3) *For cycle graph $C_n$,*

$$\mathbb{ED}(C_n,x) = \begin{cases} nx^{\frac{n^2-1}{4}}, & \text{if } n \text{ is odd;} \\ nx^{\frac{n^2}{4}}, & \text{if } n \text{ is even.} \end{cases}$$

(4) *For wheel graph $W_{n+1}, n \geq 5$,*

$$\mathbb{ED}(W_{n+1},x) = nx^{3n-3} + nx^{5n-11}.$$

*Proof.* The proof follows from Definition 2.1 and Table 3.

| Graph $G$ | $\tau_G(e), e \in E(G)$ | $\tau_j$ |
|---|---|---|
| Star $(K_{1,n})$ | $n-1$ | $n$ |
| Path $(P_n), n \geq 3$ | $\binom{n-1}{2}$, if $e_i$ is a pendant edge | 2 |
| | $\binom{i}{2} + \binom{n-i}{2}$, if $e_i$ is a nonpendant edge. | $n-3$ |
| Cycle $(C_n)$ | $\frac{n^2-1}{4}$, if $n$ is odd | $n$ |
| | $\frac{n^2}{4}$, if $n$ is even | $n$ |
| Wheel $(W_{n+1}), n \geq 5$ | $3n-3$, if $e$ is incident with universal vertex | $n$ |
| | $5n-11$, if $e$ is not incident with universal vertex. | $n$ |

Table 3

One can refer [16] for the $\tau_G(e)$.

∎

In the following proposition we provide the distance $\tau$-polynomial of some special graphs.

**Proposition 3.2.** (1) *For friendship graph* $C_3^n$ *is*

$$\mathbb{ED}(C_3^n, x) = 2nx^{4n-2} + nx^{7n-5}.$$

(2) *For helm graph* $H_n, n \geq 5$ *is*

$$\mathbb{ED}(H_n, x) = nx^{8n-12} + nx^{8n-17} + nx^{5n-4}.$$

(3) *For gear graph* $G_n, n \geq 5$ *is*

$$\mathbb{ED}(G_n, x) = 2nx^{8n-10} + nx^{5n-3}.$$

(4) *For fan graph* $F_{1,n}$, $n \geq 6$ *is*

$$\mathbb{ED}(F_{1,n}, x) = 2x^{3n-4} + 2x^{5n-11} + (n-2)x^{3n-5} + 2x^{5n-13} + (n-5)x^{5n-14}.$$

*Proof.* The proof follows from Definition 2.1 and Table 4.

| Graph $G$ | $\tau_G(e), e \in E(G)$ | $\tau_j$ |
|---|---|---|
| Friendship graph ($C_3^n$) | $4n-2$, if $e$ is incident with the universal vertex | $2n$ |
| | $7n-5$, if $e$ is not incident with the universal vertex. | $n$ |
| Helm graph $(H_n), n \geq 5$ | $8n-12$, if $e$ is pendant edge | $n$ |
| | $8n-17$, if $e$ is on the cycle $C_n$ | $n$ |
| | $5n-4$, if $e$ is incident with the central vertex. | $n$ |
| Gear graph $(G_n), n \geq 5$ | $5n-3$, if $e$ is incident with the central vertex | $n$ |
| | $8n-10$, if $e$ is not incident with the central vertex. | $2n$ |
| Fan graph $(F_{1,n}), n \geq 6$ | $3n-4$, if $e = uv$, $\deg(u) = n$ and $\deg(v) = 2$ | 2 |
| | $3n-5$, if $e = uv$, $\deg(u) = n$ and $\deg(v) = 3$ | $n-2$ |
| | $5n-11$, if $e \in E(P_n)$, $e = uv$, $deg_{F_{1,n}}(u)$ and $deg_{F_{1,n}}(v)$ | 2 |
| | $5n-13$, if $e \in E(P_n)$, $e$ is adjacent to an edge in $F_{1,3}$ of degree 3 | 2 |
| | $5n-14$, otherwise. | $n-5$ |

Table 4

One can refer [16] for the $\tau_G(e)$.

□

**Observation 3.1.** (1) *The constant term of* $\mathbb{ED}(G, x)$ *is always zero.*

(2) *Sum of all the coefficients in* $\mathbb{ED}(G, x) = |E(G)|$.

## 4. Distance $\tau$-Polynomial of some Graph Operations

In this section, the distance $\tau$-polynomial of total graphs of complete, star and cycle graphs, jump graphs of path and cycle along with complement graphs of path, cycle, helm, gear and barbell graphs are computed.

**Theorem 4.1.** *The distance $\tau$-polynomial of total graph of complete graph $K_n$, then* $\mathbb{D}(T(K_n), x) = \left(\frac{n(n+1)}{2}\right) x^{n^2-n}$.

*Proof.* The total graph of complete graph has $\frac{n(n+1)}{2}$ vertices.
Let $v \in V(G)$ then $\tau_{T(G)}(v) = (2n-2)1 + \left(\frac{n(n+1)}{2} - 2(n-1) - 1\right)2 = n^2 - n$.
Let $u \in E(G)$ then $\tau_{T(G)}(u) = (2)1+(2n-4)1+\left(\frac{n(n+1)}{2} - 2 - (2n-4) - 1\right)2 = n^2-n$.
Then,

$$\mathbb{D}(T(K_n), x) = \left(\frac{n(n+1)}{2}\right) x^{n^2-n}.$$

□

**Theorem 4.2.** *The distance $\tau$-polynomial of total graph of star graph $G = K_{1,n}$, then* $\mathbb{D}(T(K_{1,n}), x) = nx^{4n-2} + nx^{3n-1} + x^{2n}$.

*Proof.* The total graph of star graph has $2n+1$ vertices. Let $u \in V(T(G))$. If $u$ corresponds to the pendant vertex of $G$ then $\tau_{T(G)}(u) = (1)1 + (1)1 + (n-1)2 + (n-1)2 = 4n-2$. If $u$ corresponds to the edge of $G$ then $\tau_{T(G)}(u) = (n-1)1 + (2)1 + (n-1)2 = 3n-1$. If $u$ corresponds to the central vertex of $G$ then $\tau_{T(G)}(u) = 2n$.
Then,

$$\mathbb{D}(T(K_{1,n}), x) = nx^{4n-2} + nx^{3n-1} + x^{2n}.$$

□

**Theorem 4.3.** *The distance $\tau$-polynomial of total graph of cycle graph $G = C_n$, then* $\mathbb{D}(T(C_n), x) = 2nx^{\frac{n^2+n}{2}}$.

*Proof.* The total graph of cycle graph has $2n$ vertices, we consider the following cases,
**Case 1.** Let $n$ be odd and $u \in V(T(G))$ which corresponds to a vertex of $G$, Then the number of vertices with given distance from $u$ is shown in Table 5.

| $d_{T(G)}(u,v) = i$ | No. of vertices of $T(G)$ at a distance $i, 1 \le i \le \frac{n+1}{2}$ from $u$ that coresspond to | |
|---|---|---|
| | Vertices of $G$ | Edges of $G$ |
| 1 | 2 | 2 |
| 2 | 2 | 2 |
| 3 | 2 | 2 |
| 4 | 2 | 2 |
| ... | ... | ... |
| ... | ... | ... |
| $\frac{n-1}{2}$ | 2 | 2 |
| $\frac{n+1}{2}$ | 0 | 1 |

Table 5

We get, $\tau_{T(G)}(u) = 4\left(1+2+3+4+,\dots,+\frac{n-1}{2}\right) + \frac{n+1}{2} = \frac{n^2+n}{2}$.

Similarly for $u \in V(T(G))$ corresponding to an edge of $G$, we get $\tau_{T(G)}(u) = \frac{n^2+n}{2}$.

**Case 2.** Let $n$ be even and $u \in V(T(G))$ which corresponds to a vertex of $G$, Then the number of vertices with given distance from $u$ is shown in Table 6.

| $d_{T(G)}(u,v) = i$ | No. of vertices of $T(G)$ at a distance $i, 1 \le i \le \frac{n}{2}$ from $u$ that coresspond to | |
|---|---|---|
| | Vertices of $G$ | Edges of $G$ |
| 1 | 2 | 2 |
| 2 | 2 | 2 |
| 3 | 2 | 2 |
| 4 | 2 | 2 |
| ... | ... | ... |
| ... | ... | ... |
| $\frac{n}{2}$ | 1 | 2 |

Table 6

We get, $\tau_{T(G)}(u) = 4\left(1+2+3+4+,\dots,+\frac{n-2}{2}\right) + 3\left(\frac{n}{2}\right) = \frac{n^2+n}{2}$.

Similarly for $u \in V(T(G))$ corresponding to an edge of $G$, we get $\tau_{T(G)}(u) = \frac{n^2+n}{2}$.

Therefore,

$$\mathbb{D}(T(C_n), x) = 2nx^{\frac{n^2+n}{2}}.$$

□

**Theorem 4.4.** *Let $G = P_n$, $n \ge 5$, be a path graph. Then distance $\tau$-polynomial of $\overline{G}$ is $\mathbb{D}(\overline{G}, x) = 2x^n + (n-2)x^{n+1}$.*

*Proof.* Let $G = P_n$, $n \ge 5$, be a path graph. If $u \in \overline{G}$ is a pendant vertex of $G$, then $\tau_{\overline{G}}(u) = (n-2)1 + (1)2 = n$. Similarly, if $u \in \overline{G}$ is a nonpendant vertex of $G$, then $\tau_{\overline{G}}(u) = (n-3)1 + (2)2 = n+1$. Therefore, $\mathbb{D}(\overline{G}, x) = 2x^n + (n-2)x^{n+1}$. □

**Theorem 4.5.** *Let $G = C_n$, $n > 4$, be a cycle graph. Then distance $\tau$-polynomial of $\overline{G}$ is $\mathbb{D}(\overline{G}, x) = nx^{n+1}$.*

*Proof.* Let $G = C_n$, $n > 4$ be a cycle graph of order $n$. If $u \in V(\overline{G})$, then $\tau_{\overline{G}}(u) = (n-3)1 + (2)2 = (n+1)$. Then, $\mathbb{D}(\overline{G}, x) = nx^{n+1}$. □

**Theorem 4.6.** *Let $G = H_n$, $n \ge 3$, be the helm graph. Then distance $\tau$-polynomial of $\overline{G}$ is $\mathbb{D}(\overline{G}, x) = x^{3n} + nx^{2n+4} + nx^{2n+1}$.*

*Proof.* The helm graph $G = H_n$, $n \ge 3$ has $2n+1$ vertices. Let $u \in V(\overline{G})$, if $u$ corresponds to a pendant vertex of $G$ then, $\tau_{\overline{G}}(u) = (2n+1-1-1)1 + (1)2 = 2n+1$. If $u$ corresponds to a vertex of $G$ adjacent to the central vertex of $G$, then $\tau_{\overline{G}}(u) = (2n+1-5)1 + (4)2 = 2n+4$. If $u$ corresponds to the central vertex of $G$, then $\tau_{\overline{G}}(u) = (n)1 + (n)2 = 3n$. Then $\mathbb{D}(\overline{G}, x) = x^{3n} + nx^{2n+4} + nx^{2n+1}$. □

**Theorem 4.7.** *Let $G = G_n$, $n \geq 3$, be a gear graph. Then distance $\tau$-polynomial of $\overline{G}$ is* $\mathbb{D}(\overline{G}, x) = nx^{2n+3} + nx^{2n+2} + x^{3n}$.

*Proof.* The gear graph $G = G_n$, $n \geq 3$ has $2n+1$ vertices. If $u$ is a central vertex of $G$ then $\tau_{\overline{G}}(u) = (2n+1-1-n)2+n = 3n$. If $u$ is a vertex adjacent to the central vertex of $G$ then, $\tau_{\overline{G}}(u) = (2n+1-3-1)1+(3)2 = 2n+3$. If $u$ is a vertex not adjacent to the central vertex, then $\tau_{\overline{G}}(u) = (2n+1-1-2)1+(2)2 = 2n+2$. Then, $\mathbb{D}(\overline{G}, x) = nx^{2n+3}+nx^{2n+2}+x^{3n}$. □

**Theorem 4.8.** *Let $G = B_n$, $n \geq 3$, be a barbell graph. Then distance $\tau$-polynomial of $\overline{G}$ is* $\mathbb{D}(\overline{G}, x) = 2x^{3n} + (2n-2)x^{3n-2}$.

*Proof.* The barbell graph $G = B_n$, $n \geq 3$ has $2n$ vertices. If $u$ is a vertex of degree $n$ then, $\tau_{\overline{G}}(u) = (n-1)2 + (n-1)1 + (1)3 = 3n$. If $u$ is a vertex of degree $n-1$ then, $\tau_{\overline{G}}(u) = (n-1)2 + (n)1 = 3n-2$. Then, $\mathbb{D}(\overline{G}, x) = 2x^{3n} + (2n-2)x^{3n-2}$. □

**Theorem 4.9.** *Let $G = J(C_n)$, $n \geq 5$, be a jump graph of cycle graph. Then distance $\tau$-polynomial of $G$ is* $\mathbb{D}(J(C_n), x) = nx^{n+1}$.

*Proof.* If $v \in V(J(C_n))$ , then $\tau_{J(C_n)}(v) = (n-3)1 + (2)2 = (n+1)$. Then, $\mathbb{D}(J(C_n), x) = nx^{n+1}$. □

**Theorem 4.10.** *Let $G = J(P_n)$, $n \geq 5$, be a jump graph of path graph. Then distance $\tau$-polynomial of $G$ is* $\mathbb{D}(J(P_n), x) = 2x^{n-1} + (n-3)x^n$.

*Proof.* If $v \in V(J(P_n))$ of degree $n-3$, then $\tau_{J(P_n)}(v) = (n-3)1 + (1)2 = (n-1)$. If $v \in V(J(P_n))$ of degree $n-4$, then $\tau_{J(P_n)}(v) = (n-4)1 + (2)2 = n$. Then, $\mathbb{D}(J(P_n), x) = 2x^{n-1} + (n-3)x^n$. □

**Theorem 4.11.** *Let $G_1$ and $G_2$ be connected graphs of order $p$ and $n$ respectively with $x_i \in V(G_1), 1 \leq i \leq p$ and $y_j \in V(G_2), 1 \leq j \leq n$. Then*

$$\mathbb{D}(G_1 + G_2, x) = \sum_{i=1}^{p} x^{2p+n-2-d_{G_1}(x_i)} + \sum_{j=1}^{n} x^{2n+p-2-d_{G_2}(y_j)}.$$

$$\mathbb{D}(G_1 \times G_2, x) = pnx^{\{n\tau_{G_1}(x_i)+p\tau_{G_2}(y_j)\}}.$$

*Proof.* By Theorem 1.1 and Definition 2.1 we get the desired result. □

**Theorem 4.12.** *The distance $\tau$-polynomial of shadow graph $S(G)$ is given by*

$$\mathbb{D}(S(G), x) = 2nx^{\{2\tau_G(v)+2\}}.$$

*Proof.* By Theorem 1.2 and Definition 2.1 we get the desired result. □

## 5. Conclusion

In this paper, we have derived the distance $\tau$ polynomial and edge distance $\tau$ polynomial for standard graphs, total graphs, jump graphs and complementary graphs of some graphs. These findings can be extended to other graph operations and their corresponding properties can be analyzed. Future research can focus on these challenges, leading to further computations of results and properties.

## References


[1] Albina, A. and Mary, U., (2018), A study on dominator coloring of friendship and barbell graphs, International Journal of Mathematics And its Applications, 6(4), pp. 99-105.

[2] Akbari, S., Alikhani, S. and Peng, Y. H., (2010), Characterization of graphs using domination polynomials, European journal of combinatorics, 31(7), pp. 1714-1724.

[3] Alfeche, F. L., Barraza, V. and Canoy, S., (2023), Closeness centrality of vertices in graphs under some operations, European Journal of Pure and Applied Mathematics, 16(3), pp. 1406-1420.

[4] Arundhadhi, R. and Ilayarani, V., (2017), Total coloring of closed helm, flower and bistar graph family, International Journal of Scientific and Research Publications, 7(7), pp. 616-621.

[5] Balakrishnan, R. and Ranganathan, K., (2012), A textbook of graph theory, Springer Science and Business Media.

[6] Basavanagoud, B. and Sayyed, M., (2021), Hub Polynomial of a Graphs, International Journal of Applied Engineering Research, 16(3), pp. 166-173.

[7] Das, K. C. and Liu, M., (2016), Complete split graph determined by its (signless) Laplacian spectrum, Discrete Applied Mathematics, 205, pp. 45-51.

[8] Eballe, R. G., Balingit, C. M. R., Cabahug Jr, I. S., Flores, A. L. V., Lumpayao, S. M. B., Peñalosa, B. D., Tampipi, G. A. L. and Villarta, C. A., (2023), Closeness centrality in graph products, Advances and Applications in Discrete Mathematics, 39(1), pp. 29-41.

[9] Eballe, R. G., and Cabahug, I., (2021), Closeness centrality of some graph families, International Journal of Contemporary Mathematical Sciences, 16(4), pp. 127-134.

[10] Farell, E. J., (1997), A note on the clique polynomial and its relation to other graph polynomials, J. Math. Sci. Calcutta, 8, pp. 97-102.

[11] Fernau, H., Ryan, J. F. and Sugeng, K. A., (2008), A sum labelling for the generalised friendship graph, Discrete Mathematics, 308(5-6), pp. 734-740.

[12] Harary, F., (1969), Graph Theory, Addison Wesley, Reading Mass.

[13] Hosoya, H., (1988), On some counting polynomials in chemistry, Discrete applied mathematics, 19(1-3), pp. 239-257.

[14] Kulli, V. R., (2012), College Graph Theory, Vishwa Int. Publ., Gulbarga, India, 1.

[15] Maulidia, A. R., (2021), May. Elegant labeling of sun graphs and helm graphs, In Journal of Physics: Conference Series (Vol. 1872, No. 1, p. 012008). IOP Publishing.

[16] Mathad, V. and Pavithra, M., (2025), Closeness Centrality Weight and Edge Closeness Centrality Weight of Graphs, Dynamics of Continuous, Discrete and Impulsive Systems - Series B: Applications and Algorithms, 32, pp.109-124.

[17] Mathad, V. and Pavithra, M., (2025), Closeness Centrality Weight of Graphs Under some Graph Operations, Communications in Mathematics and Applications, 16(2).

[18] Meena, S. and Vaithilingam, K., (2012), Prime labeling for some fan related graphs, International Journal of Engineering Research and Technology (IJERT), 9, pp. 2278-0181.

[19] Noy, M., (2003), Graphs determined by polynomial invariants, Theoretical Computer Science, 307(2), pp. 365-384.

[20] Padmapriya, P. and Mathad, V., (2017), The eccentric-distance sum of some graphs, Electronic Journal of Graph Theory and Applications (EJGTA), 5(1), pp. 51-62.

[21] Padmapriya, P. and Mathad, V., (2020), Eccentricity based topological indices of some graphs, TWMS Journal of Applied and Engineering Mathematics, 10(4), pp. 1084-1095.

[22] Prajapati, U. M. and Raval, K. K., (2016), Product Cordial Graph in the Context of Some Graph Operations on Gear Graph, Open Journal of Discrete Mathematics, 6(4), pp. 259-267.


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