

STABILITY ANALYSIS AND DATA SENSITIVITY OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

PRABAKARAN RAGHAVENDRAN^{1,2,*}, §

ABSTRACT. The paper focuses on analyzing the Ulam stability and data dependence in fractional integro-differential equations with Caputo-type fractional derivatives. The main aim of the work is to see under which circumstances the solutions remain stably behaved with respect to small perturbations in the initial data and the parameters of the system. Using a nonlinear integral inequality of the Henry-Gronwall kind, four types of stabilities are studied and discussed. From there, several examples are shown that illustrate the theory. Also, there are some graphs to show how the system solutions behave and to depict the sensitive nature of the system concerning input data changes. The results provide valuable insights into the stability aspects of fractional dynamic systems and support existing research in fractional calculus and their applications.

Keywords: Ulam Stability, Data Interdependence, Fractional Integro-Differential Equations, Caputo Fractional Derivative, Henry-Gronwall Inequality.

AMS Subject Classification: 26A33, 26D15, 34K20, 34K40

1. INTRODUCTION

The beginnings of fractional calculus can be traced back to the challenge of extending the concept of the derivative to non-integer numerical approximations. L'Hôpital first considered this problem on September 30th, 1695. According to the study of fractional calculus has been extensively used in various fields of engineering and physics to model and analyze problems such as electromagnetics, fluid mechanics, electrochemistry, viscoelasticity, biological population modeling, nonlinear optics, nonlinear structures in plasmas, and signal processing [9]. Its relevance lies in its ability to accurately model technical and physical phenomena best characterized by fractional differential equations.

Fractional differential equations, referenced in [15], are highly effective instruments for modeling a wide range of physical phenomena, including non-uniform seismic vibrations,

¹ Center for Research, Easwari Engineering College, 18 Bharathi Salai, Ramapuram Chennai-600089, Tamil Nadu, India

² Department of Mathematics and Science Education, Faculty of Education, Harran University, Sanliurfa 63150, Turkey
e-mail: rockypraba55@gmail.com; ORCID: <https://orcid.org/0009-0001-7333-6555>.

* Corresponding Author

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fluid dynamics, traffic simulations, and nonlinear waves and oscillations in diverse plasma physics contexts.

The realm of functional equations has also received attention, particularly regarding stability characteristics such as Ulam-Hyers and Ulam-Hyers-Rassias dependencies. Works by Abbas, Benchohra, Lagreg, Alsaedi, and Zhou (2017) [1], Abdo, Thabet, and Ahmad (2020) [2], Benchohra and Lazreg (2017) [5], Salim, Benchohra, Karapinar, and Lazreg (2020) [13], and Sousa and Oliveira (2018) [16] delve into stability and existence results for various types of fractional differential equations.

Recent studies have examined the behavior of different systems utilizing fractional calculus as a framework, employing operators such as the fractional Caputo-Hadamard and the p-Laplacian operators. Kaushik et al. [10] investigated on existence of solution of the general delay of fractional differential equation with p-Laplacian operator employing principles of fixed point analysis. Later on, Gunasekar et al. [8] present the Laplace transform technique to get solutions for fractional integral-differential equations that would be more helpful in understanding a rather effective method dealing with these problems. Also, another undefined author writing in 2022 also stressed another aspect of hybrid fractional differential equations with the p-Laplacian operator, that is Hyers-Ulam stability.

The recent development in fractional differential equations concerns the existence, stability, and controllability of solutions under several nonlocal and generalized conditions. Kumar [19] applied fixed point theory to study the existence of solutions for fractional integro-differential systems with nonlocal conditions and, in so doing, laid the foundation for more elaborate models of such systems. On such foundational work, Dhawan et al. [20] studied the well-posedness and Ulam-Hyers stability of Hilfer-type fractional differential equations of higher order, further establishing the robustness of solutions under nonlocal boundary conditions. Beside this, the concept of controllability in fractional systems has been enlarged by Shukla et al. [21] via investigating approximate controllability of stochastic evolution inclusions involving Hilfer fractional derivatives, hence widening the scope of fractional calculus for uncertain dynamic environments. Shah et al. [22] have studied the dynamics of recycling materials in chemostat systems from a coupled dynamical systems perspective. Models and simulations of complex recycling processes have been made with artificial deep neural networks, focusing on how biological and chemical processes interact over time. They determined various conditions for the system to be stable and convergent; hence an invaluable approach for real-time prediction and control in environmental engineering. The authors' analysis adds substantially to the understanding of stability in nonlinear fractional-order systems and serves as a foundation for generalizations of such techniques to more general classes of evolution equations.

While fractional integro-differential equations have been considered from proper existence and stability points of view, attention has been largely paid to specific types of fractional derivatives-in particular the Hilfer or Riemann-Liouville operators [20, 21], or to one kind of Ulam stability. These works only consider a few particular boundary conditions or work under very narrow analytical tools, which means no one has really started to look into multiple stability types in a generalized set-up. Furthermore, almost every other considered almost completely theoretical works, aside from actually showing how initial data perturbations affect the behavior of solutions.

The present study provides a complete analysis of four different types of Ulam stability for fractional integro-differential equations involving the Caputo derivative, in contrast to prior studies [19, 20]. By applying a nonlinear integral inequality akin to the Henry-Gronwall inequality, conditions of stability are established in a strong and unified manner.

The detailed illustrative examples and graphical simulations not only serve to verify obtained results but also exhibit the behavior of issued trajectories concerning variations in initial conditions. This combination of rigorous analysis and insight through illustrations makes the current work very different from previous ones.

It has been noted that fractional integro-differential equations have recently attracted much attention because of the numerous application areas in science. These equations are very useful in modeling complicated phenomena with memory and heredity; consequently, they are essential in physics, engineering, and biology. However, more work needs to be done on Ulam stability results and data dependence in fractional integro-differential equations even when fractional calculus is becoming increasingly popular. To fill this research gap, this study intends to compile and systematize the existing information related to Ulam stability and incorporate the Henry-Gronwall inequality to complement the Ulam stability analysis. In the same respect, data dependence outcomes are performed with the help of the generalized Gronwall inequality that is consisted of mixed integral terms. This study aims at contributing towards the existing knowledge in the area of fractional integro-differential equations and by so doing providing new valuable information that would improve the knowledge in the field of fractional calculus. Within this discourse, we analyze the following fractional integro-differential equation:

$${}^c D^\alpha \psi(\xi) = \vartheta(\xi, \psi(\xi)) + \int_0^\xi \mu(\xi, \varsigma, \psi(\varsigma)) d\varsigma, \quad \xi \in [\beta, \varepsilon), \varepsilon = +\infty,$$

where ${}^c D^\alpha$ denotes the Caputo fractional derivative with α belonging to the open interval $(0, 1)$. We will subsequently establish specific conditions for the functions ϑ and μ . Furthermore, we investigate data dependence analysis for the following fractional integro-differential equation:

$${}^c D^\alpha \psi(\xi) = \vartheta(\xi, \psi(\xi)) + \int_0^\xi \mu(\xi, \varsigma, \psi(\varsigma)) d\varsigma, \quad \xi \in [\beta, \varepsilon), \varepsilon < +\infty,$$

where ${}^c D^\alpha$ represents the Caputo fractional derivative with α falling within the range of $(1, 2)$.

In this article, the focus lies on exploring four distinct Ulam stability definitions in the context of fractional integro-differential equations. The Henry-Gronwall inequality demonstrates these four types of Ulam stability results for fractional integro-differential equations when $0 < \alpha < 1$ and $\varepsilon = +\infty$. Additionally, attention is drawn to the intriguing data dependence results associated with the fractional integro-differential equation when $1 < \alpha < 2$ and $\varepsilon < +\infty$. This is achieved by employing a generalized Henry-Gronwall inequality incorporating a mixed integral term. To enhance our theoretical findings, an illustrative example featuring such fractional integro-differential equations is also presented.

2. PRELIMINARIES

In this section, we consider a Banach space denoted as $(\mathfrak{B}, |\cdot|)$, where β and ε are real numbers with $\beta < \varepsilon \leq +\infty$. Let $C([\beta, \varepsilon), \mathfrak{B})$ represent the Banach space comprising all continuous functions defined on the interval $[\beta, \varepsilon)$, equipped with the norm $\|\varrho\|_c = \sup\{|\varrho(\xi)| : \xi \in [\beta, \varepsilon)\}$ [12, 18, 14].

Definition 2.1. [14] The fractional integral of a function $h(\xi)$ with the Riemann-Liouville definition of order $\omega > 0$ is given by

$$J^\omega h(\xi) = \frac{1}{\Gamma(\omega)} \int_0^\xi (\xi - \zeta)^{\omega-1} h(\zeta) d\zeta, \quad \text{for } \xi > 0, \omega \in \mathbb{R}^+,$$

where \mathbb{R}^+ denotes the set of positive real numbers, and $J^0h(\xi) = h(\xi)$.

Definition 2.2. [14] The Caputo fractional derivative of order ω , where ω falls within the range of 0 to 1, is applicable to a function $h : [0, 1) \rightarrow \mathbb{R}$. It can be represented as:

$$D^\omega h(\xi) = \frac{1}{\Gamma(1 - \omega)} \int_0^\xi \frac{h'(\zeta)}{(\xi - \zeta)^\omega} d\zeta, \quad \text{for } \xi > 0.$$

Consider ϵ as a positive real number, where ϑ and μ are continuous operators mapping from the domain $[\beta, \epsilon) \times \mathfrak{B}$ to \mathfrak{B} , while ρ is a continuous function mapping from the domain $[\beta, \epsilon)$ to the set of positive real numbers, denoted as \mathbb{R}^+ . Let $\psi : [\beta, \epsilon) \rightarrow \mathfrak{B}$ be a continuous function. Now, consider the following integro-differential equation

$${}^c D^\alpha \psi(\xi) = \vartheta(\xi, \psi(\xi)) + \int_\beta^\xi \mu(\xi, \varsigma, \psi(\varsigma)) d\varsigma, \quad \alpha \in (0, 1) \text{ (or } (1, 2)), \xi \in [\beta, \epsilon). \quad (1)$$

We examine the corresponding inequality for an approximate function $\varrho(\xi)$, which is assumed to be a close approximation to the exact solution $\psi(\xi)$:

$$|{}^c D^\alpha \varrho(\xi) - \vartheta(\xi, \varrho(\xi)) - \int_\beta^\xi \mu(\xi, \varsigma, \varrho(\varsigma)) d\varsigma| \leq \epsilon, \quad \xi \in [\beta, \epsilon), \quad (2)$$

$$|{}^c D^\alpha \varrho(\xi) - \vartheta(\xi, \varrho(\xi)) - \int_\beta^\xi \mu(\xi, \varsigma, \varrho(\varsigma)) d\varsigma| \leq \rho(\xi), \quad \xi \in [\beta, \epsilon), \quad (3)$$

$$|{}^c D^\alpha \varrho(\xi) - \vartheta(\xi, \varrho(\xi)) - \int_\beta^\xi \mu(\xi, \varsigma, \varrho(\varsigma)) d\varsigma| \leq \epsilon \rho(\xi), \quad \xi \in [\beta, \epsilon). \quad (4)$$

The equation (1) is considered U-H stable if there exists a positive real number $c_{\vartheta, \mu}$ such that, for any given $\epsilon > 0$ and for every solution ϱ in either $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) of the inequality (2), there exists a solution ψ in either $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) of the equation (1) such that $|\varrho(\xi) - \psi(\xi)|$ is less than or equal to $c_{\vartheta, \mu} \epsilon$ for all ξ in the interval $[\beta, \epsilon)$.

The equation (1) is considered generalized U-H stable if there exists a positive real number $\theta_{\vartheta, \mu}$ such that, for any given $\epsilon > 0$ and for every solution ϱ in either $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) of the inequality (2), there exists a solution ψ in either $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) of the equation (1) such that $|\varrho(\xi) - \psi(\xi)|$ is less than or equal to $\theta_{\vartheta, \mu}(\epsilon)$ for all ξ in the interval $[\beta, \epsilon)$.

The equation (1) demonstrates U-H-R stability with respect to ρ if there exists a positive real number $c_{\vartheta, \mu, \rho}$ such that, for any given $\epsilon > 0$ and for each solution ϱ in either $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) of the inequality (4), there exists a solution ψ in either $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) of the equation (1) such that $|\varrho(\xi) - \psi(\xi)|$ is less than or equal to $c_{\vartheta, \mu, \rho} \epsilon \rho(\xi)$ for all ξ in the interval $[\beta, \epsilon)$.

The equation (1) demonstrates generalized U-H-R stability with respect to ρ if there exists a positive real number $c_{\vartheta, \mu, \rho}$ such that, for any given $\epsilon > 0$ and for each solution ϱ in either $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) of the inequality (3), there exists a solution ψ in either $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) of the equation (1) such that $|\varrho(\xi) - \psi(\xi)|$ is less than or equal to $c_{\vartheta, \mu, \rho} \epsilon \rho(\xi)$ for all ξ in the interval $[\beta, \epsilon)$.

Remark 2.1. A function ϱ in the space $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) is considered a solution to the inequality (2) if and only if there exists another function g defined on the interval $[\beta, \epsilon)$ and belonging to either $C^1([\beta, \epsilon), \mathfrak{B})$ (or $C^2([\beta, \epsilon), \mathfrak{B})$) such that:

$$(1) |g(\xi)| \leq \epsilon, \text{ for all } \xi \in [\beta, \epsilon).$$

(2) The function ψ satisfies the equation

$${}^c D^\alpha \psi(\xi) = \vartheta(\xi, \psi(\xi)) + \int_0^\xi \mu(\xi, \varsigma, \psi(\varsigma)) d\varsigma + g(\xi), \quad \xi \in [\beta, \varepsilon].$$

Similar arguments can be applied to inequalities (3) and (4). Therefore, Ulam stability in fractional integro-differential equations can be understood as a specific manifestation of data dependence exhibited by the solutions of such equations.

Remark 2.2. For any α satisfying $0 < \alpha < 1$, if a function ϱ belongs to the class $C^1([\beta, \varepsilon], \mathfrak{B})$ and satisfies the inequality (2), then it also serves as a solution to the following integral inequality:

$$\left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \varrho(\varsigma)) + \int_\beta^\varsigma \mu(\sigma, \varsigma, \varrho(\varsigma)) d\sigma \right] d\varsigma \right| \leq \frac{(\xi - \beta)^\alpha \epsilon}{\Gamma(\alpha + 1)},$$

$$\xi \in [\beta, \varepsilon].$$

By the previous result, we have:

$${}^c D^\alpha \varrho(\xi) = \vartheta(\xi, \varrho(\xi)) + \int_\beta^\xi \mu(\xi, \varsigma, \varrho(\varsigma)) d\varsigma + g(\xi), \quad \forall \xi \in [\beta, \varepsilon].$$

Applying the integral operator yields:

$$\begin{aligned} \varrho(\xi) - \varrho(\beta) &= \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} \vartheta(\varsigma, \varrho(\varsigma)) d\varsigma + \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} \\ &\quad \left(\int_\beta^\varsigma \mu(\sigma, \varsigma, \varrho(\varsigma)) d\sigma \right) d\varsigma + \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} g(\varsigma) d\varsigma. \end{aligned}$$

Thus:

$$\begin{aligned} \left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \varrho(\varsigma)) + \int_\beta^\varsigma \mu(\sigma, \varsigma, \varrho(\varsigma)) d\sigma \right] d\varsigma \right| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} g(\varsigma) d\varsigma \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} |g(\varsigma)| d\varsigma \\ &\leq \frac{\epsilon}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} d\varsigma \\ &\leq \frac{(\xi - \beta)^\alpha \epsilon}{\Gamma(\alpha + 1)}. \end{aligned}$$

Similar remarks can be made about inequalities (3) and (4).

Lemma 2.1. [18] Let ϱ and η be continuous functions mapping from $[0, T)$ to $[0, +\infty)$, where $T \leq \infty$. Suppose η is non-decreasing and there exist constants $m \leq 0$ and $n > 0$ such that:

$$\varrho(\xi) \leq \eta(\xi) + m \int_0^\xi (\xi - \varsigma)^{n-1} \varrho(\varsigma) d\varsigma, \quad \xi \in [0, T).$$

Then,

$$\varrho(\xi) \leq \eta(\xi) + \int_0^\xi \left[\sum_{r=0}^{\infty} \frac{(m\Gamma(n))^r}{\Gamma(rn)} (\xi - \varsigma)^{rn-1} \eta(\varsigma) \right] d\varsigma, \quad \xi \in [0, T).$$

If $\eta(\xi) = \bar{\beta}$ is a constant function on the interval $0 \leq \xi < T$, the inequality simplifies to:

$$\varrho(\xi) \leq \bar{\beta} E_n(m\Gamma(n)t^n), \quad 0 \leq t < T,$$

where E_n denotes the Mittag-Leffler function defined by:

$$E_{\sigma,\tau}(v) = \sum_{\gamma=0}^{\infty} \frac{v^\gamma}{\Gamma(\sigma\gamma + \tau)}, \quad (v, \sigma, \tau \in \mathbb{C}, \mathbb{R} \text{ with } \sigma > 0).$$

Remark 2.3. [18] Consider the case where $q < +\infty$ and ϱ is a continuous function defined on the interval $[0, q]$ with values in the Banach space \mathfrak{B} . Suppose ϱ satisfies the inequality:

$$|\varrho(\xi)| \leq p_1 + q_1 \int_0^\xi (\xi - \varsigma)^{\delta-1} |\varrho(\varsigma)|^\gamma d\varsigma + r_1 \int_0^q (q - \varsigma)^{\alpha-1} |\varrho(\varsigma)|^\gamma d\varsigma,$$

where $\delta \in (1, 2)$, $\gamma \in [0, 1 - \frac{1}{m}]$ for some $1 < m < +\infty$, and $p_1, q_1, r_1 \geq 0$ are constants. If there exists a constant $M := (q_1 + r_1) \left[\frac{q^{m(\delta-1)+1}}{p_1^{(\delta-1)+1}} \right]^{\frac{1}{m}} > 0$, then $|\varrho(\xi)| \leq (p_1 + 1)e^{Mq}$.

3. ULAM STABILITY RESULTS AND DATA DEPENDENCE

In this section, we examine Ulam stability in fractional integro-differential equations, focusing on how stability depends on variations in data. We present theoretical results and discuss their implications for the stability of solutions.

3.1. Ulam Stability Results. In this subsection, we investigate the Ulam stability of fractional integro-differential equations. We detail the conditions under which solutions exhibit stability, focusing on the influence of data variations on the behavior of solutions.

Consider the fractional integro-differential equation:

$${}^c D^\alpha \psi(\xi) = \vartheta(\xi, \psi(\xi)) + \int_\beta^\xi \mu(\xi, \varsigma, \psi(\varsigma)) d\varsigma; \quad \alpha \in (0, 1) \text{ (or } (1, 2)), \quad \xi \in [\beta, \varepsilon].$$

Where $\psi(\xi)$ is the unknown solution. Let $\varrho(\xi)$ be a continuous approximate function. We analyze its deviation from the exact solution via the inequality:

$$\left| {}^c D^\alpha \varrho(\xi) - \left[\vartheta(\xi, \varrho(\xi)) + \int_\beta^\xi \mu(\xi, \varsigma, \varrho(\varsigma)) d\varsigma \right] \right| \leq \phi(\xi); \quad \xi \in [\beta, \varepsilon].$$

This analysis is conducted under the assumption that $\varepsilon = +\infty$. We assume the following conditions:

(H₁) $k, \vartheta \in C([\beta, \infty) \times \mathfrak{B}, \mathfrak{B})$.

(H₂) There exist constants $m_{1\vartheta}, m_{2\mu} > 0$ such that:

$$|\vartheta(\xi, u_1) - \vartheta(\xi, u_2)| \leq m_{1\vartheta}|u_1 - u_2|,$$

$$|\mu(\xi, \varsigma, v_1) - \mu(\xi, \varsigma, v_2)| \leq m_{2\mu}|v_1 - v_2|,$$

for all $\xi \in [\beta, \varepsilon)$ and $u_1, u_2, v_1, v_2 \in \mathfrak{B}$.

(H₃) If ρ is an increasing function in $C([\beta, +\infty), \mathbb{R}_+)$, then there exists a positive constant $\lambda_\rho > 0$ such that:

$$\frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} \rho(\varsigma) d\varsigma \leq \lambda_\rho \rho(\xi) \quad \text{for each } \xi \in [\beta, \varepsilon].$$

Under these conditions, we obtain the following generalized Ulam-Hyers-Rassias stability results.

Theorem 3.1. Under the conditions (H1), (H2), and (H3), the equation (1) with $\varepsilon = +\infty$ is generally classified as U-H-R stable.

Proof. Consider $\varrho \in C^1([a, +\infty), \mathfrak{B})$ satisfying the inequality (3) with $\varepsilon = +\infty$. Let ψ be the unique solution of the Cauchy problem:

$${}^c D^\alpha \psi(\xi) = \vartheta(\xi, \psi(\xi)) + \int_{\beta}^{\xi} \mu(\xi, \varsigma, \psi(\varsigma)) d\varsigma, \quad 0 < \alpha < 1, \quad \xi \in [\beta, \varepsilon), \quad \text{with } \psi(\beta) = \varrho(\beta), \quad (5)$$

where $\psi(\xi)$ is the unknown function and ϱ is a given continuous initial function. Then we have:

$$\psi(\xi) = \varrho(\beta) + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \psi(\varsigma)) + \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \psi(\sigma)) d\sigma \right] d\varsigma.$$

Applying the integral operator on inequality (3), we get

$$\varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \vartheta(\varsigma, \varrho(\varsigma)) d\varsigma + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} + \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma d\varsigma$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \rho(\varsigma) d\varsigma$$

$$\begin{aligned} & \left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \varrho(\varsigma)) + \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma \right] d\varsigma \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} |\rho(\varsigma)| d\varsigma \\ & \leq \lambda_{\rho} \rho(\xi); \quad \xi \in [a, +\infty). \end{aligned}$$

It follows,

$$\begin{aligned}
 \left| \varrho(\xi) - \psi(\xi) \right| &\leq \left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \psi(\varsigma)) + \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \psi(\sigma)) d\sigma \right] d\varsigma \right| \\
 &\leq \left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \vartheta(\varsigma, \psi(\varsigma)) d\varsigma \right. \\
 &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \psi(\sigma)) d\sigma d\varsigma \right| \\
 &\leq \left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \vartheta(\varsigma, \varrho(\varsigma)) d\varsigma + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \vartheta(\varsigma, \varrho(\varsigma)) d\varsigma \right. \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \vartheta(\varsigma, \psi(\varsigma)) d\varsigma - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma d\varsigma \\
 &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\varsigma}^{\xi} \mu(\sigma, \varsigma, \varrho(\varsigma)) d\sigma d\varsigma - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \psi(\sigma)) d\sigma d\varsigma \right| \\
 &\leq \left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \vartheta(\varsigma, \varrho(\varsigma)) d\varsigma \right. \\
 &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma d\varsigma \right| \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left| \vartheta(\varsigma, \varrho(\varsigma)) - \vartheta(\varsigma, \psi(\varsigma)) \right| d\varsigma
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\beta}^{\varsigma} \left| \mu(\sigma, \varsigma, \varrho(\varsigma)) - \mu(\sigma, \varsigma, \psi(\varsigma)) \right| d\sigma d\varsigma \\
 &\leq \left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \varrho(\varsigma)) + \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma \right] d\varsigma \right|
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} m_{1\vartheta} |\varrho(\varsigma) - \psi(\varsigma)| d\varsigma + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} m_{2\mu} |\varrho(\varsigma) - \psi(\varsigma)| d\varsigma \\
& \leq \lambda_{\rho} \rho(\xi) + \frac{m_{1\vartheta}}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} |\varrho(\varsigma) - \psi(\varsigma)| d\varsigma \\
& + \frac{m_{2\mu}}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} |\varrho(\varsigma) - \psi(\varsigma)| (\xi - \varsigma) d\varsigma \\
& \leq \lambda_{\rho} \rho(\xi) + \frac{m_{1\vartheta}}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} |\varrho(\varsigma) - \psi(\varsigma)| d\varsigma + \frac{m_{2\mu}}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha} |\varrho(\varsigma) - \psi(\varsigma)| d\varsigma \\
& \leq \lambda_{\rho} \rho(\xi) + \frac{m_{1\vartheta}}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} |\varrho(\varsigma) - \psi(\varsigma)| d\varsigma + \frac{m_{2\mu}}{\Gamma(\alpha)} |\varrho(\varsigma) - \psi(\varsigma)| \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha} d\varsigma \\
& \leq \lambda_{\rho} \rho(\xi) + \frac{m_{1\vartheta}}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} |\varrho(\varsigma) - \psi(\varsigma)| d\varsigma + \frac{m_{2\mu}}{\Gamma(\alpha)} |\varrho(\varsigma) - \psi(\varsigma)| \frac{T^{\alpha+1}}{\alpha+1} \\
& \leq \lambda_{\rho} \rho(\xi) + \frac{m_{1\vartheta}}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} |\varrho(\varsigma) - \psi(\varsigma)| d\varsigma + \frac{m_{2\mu}}{\Gamma(\alpha)(\alpha+1)} |\varrho(\varsigma) - \psi(\varsigma)| T^{\alpha+1} \\
& \leq \lambda_{\rho} \rho(\xi) + \frac{m_{1\vartheta}}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} |\varrho(\varsigma) - \psi(\varsigma)| d\varsigma + m_{2\mu} |\varrho(\varsigma) - \psi(\varsigma)| \frac{\alpha T^{\alpha+1}}{(\alpha+1) \frac{\Gamma(\alpha+1)}{\alpha}}.
\end{aligned}$$

By applying Lemma 2.1 and Remark 2.1, there exists a constant $M_{\vartheta, \mu}^* > 0$ that is independent of $\lambda_{\rho} \rho(\xi)$, such that:

$$|\varrho(\xi) - \psi(\xi)| \leq M_{\vartheta, \mu}^* \lambda_{\rho} \rho(\xi), \quad \xi \in [a, +\infty).$$

Therefore, the equation (1) is classified as exhibiting generalized Ulam-Hyers-Rassias stability.

3.2. Data Dependence. In this subsection, we examine how variations in data affect the stability of fractional integro-differential equations. We analyze the sensitivity of solutions to changes in input data, highlighting the robustness and adaptability of the models under different conditions.

For $1 < \alpha < 2$, we consider the following equation:

$${}^c D^{\alpha} \psi(\xi) = \vartheta(\xi, \psi(\xi)) + \int_{\beta}^{\xi} \mu(\xi, \varsigma, \psi(\varsigma)) d\varsigma \quad \alpha \in (0, 1) \text{ or } (1, 2); \xi \in [\beta, \varepsilon),$$

and the inequality:

$$\left| {}^c D^{\alpha} \varrho(\xi) - \left[\vartheta(\varsigma, \varrho(\varsigma)) + \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma d\varsigma \right] \right| \leq \varepsilon; \quad \xi \in [\beta, \varepsilon),$$

in the case $\varepsilon < +\infty$. We assume the following conditions:

- (H4) $\vartheta, \mu \in C([\beta, \varepsilon] \times \mathfrak{B})$.
- (H5) There exist positive constants $m_{\vartheta, \mu} > 0$ and $\lambda \in [0, 1 - \frac{1}{p}]$, where p is a real number with $1 < p < \infty$, such that:

$$|\vartheta(\xi, u_1) - \vartheta(\xi, u_2)| \leq m_{1\vartheta} |u_1 - u_2|^{\lambda},$$

$$|\mu(\sigma, \varsigma, v_1) - \mu(\sigma, \varsigma, v_2)| \leq m_{2\mu} |v_1 - v_2|^{\lambda},$$

for all $\xi \in [\beta, \varepsilon)$ and all $u_1, u_2, v_1, v_2 \in \mathfrak{B}$.

Theorem 3.2. Assume that the assumptions (H4) and (H5) hold. Let $\varrho \in C^2[\beta, \varepsilon]$ satisfy inequality (2). Let ψ denote the solution to the following boundary value problem:

$${}^c D^\alpha \psi(\xi) = \vartheta(\xi, \psi(\xi)) + \int_\beta^\xi \mu(\xi, \varsigma, \psi(\varsigma)) d\varsigma, \quad 1 < \alpha < 2, \quad \xi \in [\beta, \varepsilon], \quad (6)$$

with boundary conditions $\psi(\beta) = \varrho(\beta)$ and $\psi(\varepsilon) = \varrho(\varepsilon)$. Then, the following inequality holds:

$$|\varrho(\xi) - \psi(\xi)| \leq c_{\vartheta, \mu}(\varepsilon + 1), \quad \xi \in [\beta, \varepsilon], \quad (7)$$

where

$$c_{\vartheta, \mu} = e^{Mb} \max \left[\frac{(\beta - \varepsilon)^\alpha}{\Gamma(\alpha + 1)}, 1 \right] > 0,$$

and

$$M = \frac{2m_{\vartheta, \mu}}{\Gamma(\alpha)} \left[\frac{b^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right]^{\frac{1}{p}}.$$

Proof: By Lemma (3.1) of [3], the solution $\psi(\xi)$ is given by:

$$\begin{aligned} \psi(\xi) &= \frac{\varepsilon - \xi}{\varepsilon - \beta} \varrho(\beta) + \frac{\xi - \beta}{\varepsilon - \beta} \varrho(\varepsilon) + \frac{\beta - \xi}{\varepsilon - \beta} \frac{1}{\Gamma(\alpha)} \int_\beta^\varepsilon (\xi - \varsigma)^{\alpha-1} \\ &\quad \left[\vartheta(\varsigma, \psi(\varsigma)) + \int_\beta^\varsigma \mu(\varsigma, \sigma, \psi(\sigma)) d\sigma \right] d\varsigma \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \psi(\varsigma)) + \int_\beta^\varsigma \mu(\varsigma, \sigma, \psi(\sigma)) d\sigma \right] d\varsigma. \end{aligned}$$

Using the differential inequality (2), we obtain:

$$\begin{aligned} \varrho(\xi) &= \frac{b - \xi}{\varepsilon - \beta} \varrho(\beta) + \frac{\xi - \beta}{\varepsilon - \beta} \varrho(\varepsilon) + \frac{\beta - \xi}{\varepsilon - \beta} \frac{1}{\Gamma(\alpha)} \int_\beta^\varepsilon (\xi - \varsigma)^{\alpha-1} \\ &\quad \left[\vartheta(\varsigma, \varrho(\varsigma)) + \int_\beta^\varsigma \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma \right] d\varsigma \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \varrho(\varsigma)) + \int_\beta^\varsigma \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma \right] d\varsigma. \end{aligned}$$

Thus, we have:

$$\begin{aligned} \left| \varrho(\xi) - \frac{b - \xi}{\varepsilon - \beta} \varrho(\beta) - \frac{\xi - \beta}{\varepsilon - \beta} \varrho(\varepsilon) - \frac{\beta - \xi}{\varepsilon - \beta} \frac{1}{\Gamma(\alpha)} \int_\beta^\varepsilon (\xi - \varsigma)^{\alpha-1} \right. \\ \left. \left[\vartheta(\varsigma, \varrho(\varsigma)) + \int_\beta^\varsigma \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma \right] d\varsigma - \frac{1}{\Gamma(\alpha)} \int_\beta^\xi (\xi - \varsigma)^{\alpha-1} \right. \\ \left. \left[\vartheta(\varsigma, \varrho(\varsigma)) + \int_\beta^\varsigma \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma \right] d\varsigma \right| \leq \frac{\epsilon}{\Gamma(\alpha)} \int_\beta^\varepsilon (\xi - \varsigma)^{\alpha-1} d\varsigma \\ \leq \frac{(\varepsilon - \beta)^\alpha \epsilon}{\alpha \Gamma(\alpha)} = \frac{(\varepsilon - \beta)^\alpha \epsilon}{\Gamma(\alpha + 1)}. \end{aligned}$$

Thus, we have:

$$\begin{aligned} |\varrho(\xi) - \psi(\xi)| &\leq \left| \varrho(\xi) - \frac{b-\xi}{\varepsilon-\beta} \varrho(\beta) - \frac{\xi-\beta}{\varepsilon-\beta} \varrho(\varepsilon) \right. \\ &\quad \left. - \frac{\beta-\xi}{\varepsilon-\beta} \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\varepsilon} (\xi-\varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \psi(\varsigma)) + \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \psi(\sigma)) d\sigma \right] d\varsigma \right| \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi-\varsigma)^{\alpha-1} \left[\vartheta(\varsigma, \psi(\varsigma)) + \int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \psi(\sigma)) d\sigma \right] d\varsigma \right| \\ &\leq \frac{(\beta-\varepsilon)^{\alpha} \varepsilon}{\Gamma(\alpha+1)} + \frac{|\beta-\xi|}{\varepsilon-\beta} \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\varepsilon} (\varepsilon-\varsigma)^{\alpha-1} |\vartheta(\varsigma, \varrho(\varsigma)) - \vartheta(\varsigma, \psi(\varsigma))| d\varsigma \\ &\quad + \frac{|\beta-\xi|}{\varepsilon-\beta} \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\varepsilon} (\varepsilon-\varsigma)^{\alpha-1} \int_{\beta}^{\xi} |\mu(\sigma, \varsigma, \varrho(\varsigma)) - \mu(\sigma, \varsigma, \psi(\varsigma))| d\sigma d\varsigma. \end{aligned}$$

Thus, we obtain:

$$|\varrho(\xi) - \psi(\xi)| \leq c_{\vartheta, \mu}(\varepsilon + 1),$$

where:

$$c_{\vartheta, \mu} = e^{Mb} \max \left[\frac{(\beta-\varepsilon)^{\alpha}}{\Gamma(\alpha+1)}, 1 \right] > 0,$$

and

$$M = \frac{2m_{\vartheta, \mu}}{\Gamma(\alpha)} \left[\frac{b^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right]^{\frac{1}{p}}.$$

Substituting the Remark (2.3) into the above equation, we get the inequality (7).

Proposition 3.1. Consider the case where $0 < \alpha < 1$ and $\mathfrak{B} = \mathbb{R}$. We analyze the following equation:

$${}^c D^{\alpha} \psi(\xi) = \int_{\beta}^{\xi} \mu(\xi, \varsigma, \psi(\varsigma)) d\varsigma \quad ; \quad \xi \in [\beta, \varepsilon], \quad (8)$$

and the associated inequality:

$$\left| {}^c D^{\alpha} \varrho(\xi) - \int_{\beta}^{\xi} \mu(\xi, \varsigma, \varrho(\varsigma)) d\varsigma \right| \leq \varepsilon \quad ; \quad \xi \in [\beta, \varepsilon]. \quad (9)$$

If ϱ is a solution to the inequality (9) on the interval $[\beta, \varepsilon)$, then there exists a function g in $C[\beta, \varepsilon)$ such that:

$$\begin{aligned} (i) \quad &|g(\xi)| \leq \varepsilon \quad ; \quad \xi \in [\beta, \varepsilon), \\ (ii) \quad &{}^c D^{\alpha} \psi(\xi) = \int_{\beta}^{\xi} \mu(\xi, \varsigma, \varrho(\varsigma)) d\varsigma + g(\xi) \quad ; \quad \xi \in [\beta, \varepsilon). \end{aligned} \quad (10)$$

Integrating (10) from β to ξ , we obtain:

$$\varrho(\xi) - \varrho(\beta) = \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi-\varsigma)^{\alpha-1} \left(\int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma \right) d\varsigma + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi-\beta)^{\alpha-1} g(\varsigma) d\varsigma.$$

$$\varrho(\xi) = \varrho(\beta) + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi-\varsigma)^{\alpha-1} \left(\int_{\beta}^{\varsigma} \mu(\varsigma, \sigma, \varrho(\sigma)) d\sigma \right) d\varsigma + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi-\beta)^{\alpha-1} g(\varsigma) d\varsigma.$$

For any $c \in \mathbb{R}$, we have:

$$\begin{aligned} |\varrho(\xi) - c| &= \left| \varrho(\beta) + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left(\int_{\beta}^{\varsigma} |\mu(\varsigma, \sigma, \varrho(\sigma))| d\sigma \right) d\varsigma \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \beta)^{\alpha-1} |g(\varsigma)| d\varsigma - c \right| \\ &\leq |\varrho(\beta) - c| + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left| \int_{\beta}^{\varsigma} |\mu(\varsigma, \sigma, \varrho(\sigma))| d\sigma \right| d\varsigma + \frac{\epsilon}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \beta)^{\alpha-1} d\varsigma \\ &\leq |\varrho(\beta) - c| + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left| \int_{\beta}^{\varsigma} |\mu(\varsigma, \sigma, \varrho(\sigma))| d\sigma \right| d\varsigma + \frac{\epsilon(\xi - \beta)^{\alpha}}{\Gamma(\alpha)}. \end{aligned}$$

By setting $c = \varrho(\beta)$, we get:

$$\begin{aligned} |\varrho(\xi) - \varrho(\beta)| &\leq \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left| \int_{\beta}^{\varsigma} |\mu(\varsigma, \sigma, \varrho(\sigma))| d\sigma \right| d\varsigma + \frac{\epsilon(\xi - \beta)^{\alpha}}{\Gamma(\alpha)} \\ \left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\beta}^{\varsigma} |\mu(\varsigma, \sigma, \varrho(\sigma))| d\sigma \right| &\leq \frac{\epsilon(\xi - \beta)^{\alpha}}{\Gamma(\alpha)}. \end{aligned}$$

If $\epsilon < +\infty$, then:

$$\left| \varrho(\xi) - \varrho(\beta) - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\beta}^{\varsigma} |\mu(\varsigma, \sigma, \varrho(\sigma))| d\sigma \right| \leq \frac{\epsilon(\xi - \beta)^{\alpha}}{\Gamma(\alpha)}. \tag{11}$$

Thus, the equation (8) is U-H stable.

If $\epsilon = +\infty$, the function

$$\varrho(\xi) = \frac{(\xi - \beta)^{\alpha} \epsilon}{\Gamma(\alpha + 1)},$$

satisfies the inequality (9), and

$$\begin{aligned} \left| \varrho(\xi) - c - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\beta}^{\varsigma} |\mu(\varsigma, \sigma, \varrho(\sigma))| d\sigma \right| \\ = \left| \frac{(\xi - \beta)^{\alpha} \epsilon}{\Gamma(\alpha + 1)} - c - \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \int_{\beta}^{\varsigma} |\mu(\varsigma, \sigma, \varrho(\sigma))| d\sigma \right| \\ \rightarrow \infty \quad \text{as } \xi \rightarrow \infty. \end{aligned}$$

Therefore, the equation (8) is U-H stable on the interval $[\beta, +\infty)$.

Now, consider the inequality:

$${}^c D^{\alpha} \varrho(\xi) \leq \int_{\beta}^{\xi} \mu(\xi, \varsigma, \varrho(\varsigma)) d\varsigma + \epsilon. \tag{12}$$

By solving (12) and integrating, we get:

$$\varrho(\xi) \leq \varrho(\beta) + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \left(\int_{\beta}^{\varsigma} |\mu(\varsigma, \sigma, \varrho(\sigma))| d\sigma \right) d\varsigma + \frac{\epsilon(\xi - \beta)^{\alpha}}{\Gamma(\alpha)}. \tag{13}$$

Thus, the equation (8) is U-H stable for $\xi \in [\beta, \epsilon)$.

If $\epsilon = +\infty$, the stability result also holds for $\xi \in [\beta, +\infty)$, as shown by the solution of (14).

Example 3.1. Consider the case where $0 < \alpha < 1$ and $\mathfrak{B} = \mathbb{R}$. We examine the following equation:

$${}^c D^\alpha \psi(\xi) = 0; \quad \xi \in [\beta, \varepsilon), \quad (14)$$

and the inequality:

$$|{}^c D^\alpha \varrho(\xi)| \leq \epsilon; \quad \xi \in [\beta, \varepsilon). \quad (15)$$

Let $\varrho \in C^1[\beta, \varepsilon)$ be a solution to equation (15). Then there exists a function $g \in C[\beta, \varepsilon)$ such that:

$$\begin{aligned} (i) \quad & |g(\xi)| \leq \epsilon; \quad \xi \in [\beta, \varepsilon), \\ (ii) \quad & {}^c D^\alpha \psi(\xi) = g(\xi); \quad \xi \in [\beta, \varepsilon). \end{aligned} \quad (16)$$

Integrating (16) from β to ξ , we obtain:

$$\varrho(\xi) = \varrho(\beta) + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - s)^{\alpha-1} g(s) ds; \quad \xi \in [\beta, \varepsilon).$$

For any $c \in \mathbb{R}$, we have:

$$\begin{aligned} |\varrho(\xi) - c| &= \left| \varrho(\beta) + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - s)^{\alpha-1} g(s) ds - c \right| \\ &\leq |\varrho(\beta) - c| + \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - s)^{\alpha-1} |g(s)| ds \\ &\leq |\varrho(\beta) - c| + \frac{\epsilon}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - s)^{\alpha-1} ds \\ &\leq |\varrho(\beta) - c| + \frac{\epsilon(\xi - \beta)^\alpha}{\Gamma(\alpha + 1)}. \end{aligned}$$

Setting $c = \varrho(\beta)$, we obtain:

$$|\varrho(\xi) - \varrho(\beta)| \leq \frac{\epsilon(\xi - \beta)^\alpha}{\Gamma(\alpha + 1)}, \quad \xi \in [\beta, \varepsilon).$$

If $\varepsilon < +\infty$, then:

$$|\varrho(\xi) - \varrho(\beta)| \leq \frac{\epsilon(\varepsilon - \xi)^\alpha}{\Gamma(\alpha + 1)}, \quad \xi \in [\beta, \varepsilon).$$

Hence, equation (14) exhibits U-H stability.

If $\varepsilon = +\infty$, the function:

$$\varrho(\xi) = \frac{(\xi - \beta)^\alpha \epsilon}{\Gamma(\alpha + 1)},$$

satisfies inequality (15) and:

$$|\varrho(\xi) - c| = \left| \frac{(\xi - \beta)^\alpha \epsilon}{\Gamma(\alpha + 1)} - c \right| \rightarrow \infty \quad \text{as } \xi \rightarrow \infty.$$

Therefore, equation (14) is U-H stable on the interval $[\beta, +\infty)$.

Now, consider the inequality:

$$|{}^c D^\alpha \varrho(\xi)| \leq \rho(\xi); \quad \xi \in [\beta, +\infty). \quad (17)$$

Let ϱ be a solution to equation (14), and $\psi(\xi) = \varrho(\beta)$ for $\xi \in [\beta, +\infty)$. We have:

$$|\varrho(\xi) - \psi(\xi)| = |\varrho(\xi) - \varrho(\beta)| \leq \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \rho(\varsigma) d\varsigma; \quad \xi \in [\beta, +\infty).$$

If there exists a constant $c_{\rho} > 0$ such that:

$$\frac{1}{\Gamma(\alpha)} \int_{\beta}^{\xi} (\xi - \varsigma)^{\alpha-1} \rho(\varsigma) d\varsigma \leq c_{\rho} \rho(\xi); \quad \xi \in [\beta, +\infty),$$

then, under these conditions, it is evident that equation (14) demonstrates generalized U-H-R stability for ρ on the interval $[\beta, +\infty)$.

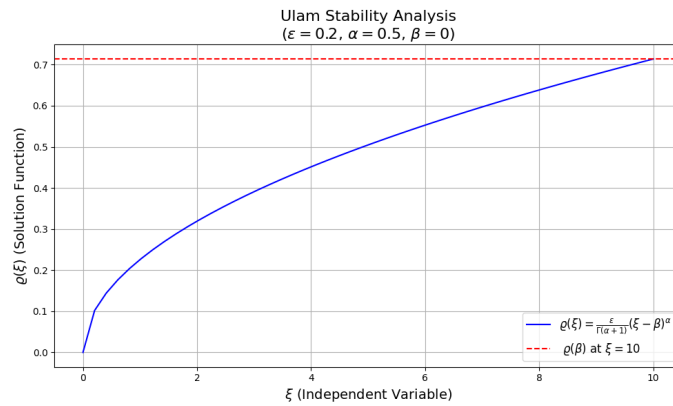


FIGURE 1. Plot of $\varrho(\xi)$ as a function of ξ for $\beta = 0$, $\epsilon = 1$, and $\alpha = 0.5$.

The 2D plot in Figure 1 illustrates the behavior of the function $\varrho(\xi)$ over the range from β to 10, parameterized by α and ϵ . The curve represents the fractional-order behavior of $\varrho(\xi)$ as ξ varies, demonstrating how the function evolves under the influence of these parameters. The red dashed horizontal line denotes the constant value $\varrho(\beta)$ at $\xi = 10$, serving as a reference point for comparison. As ξ increases, the distance between the curve of $\varrho(\xi)$ and the horizontal line provides insight into the stability characteristics of the system. Specifically, the graph illustrates how $\varrho(\xi)$ transitions relative to $\varrho(\beta)$, highlighting the impact of fractional order α on stability. This visualization helps in the analysis of the systems Ulam stability properties, and its depiction on the function and the initial condition as well as the fractional differential operator.

In Figure 2, there is a 3D surface plot of $\varrho(\xi)$ against ξ and another parameter φ . This graph shows the level of $\varrho(\xi)$ with respect to different values of ξ and φ ; bright colors on the surface represent high values of $\varrho(\xi)$, while low values fade. A colourful grid is useful for visualizing how this value varies with both parameters, thus drawing attention to changes in the functions values embodied in $\varrho(\xi)$. The surface plot gives a gist showing the interaction of the data sets ξ and φ with $\varrho(\xi)$ making it easy to predict the high and low regions of data dependence. The grid lines and the colour gradient also clarify the functions responsiveness to changes in a parameter and the patterns within the data points. This kind of 3D representation is more helpful in observing the functions performance under certain circumstances and to get a better look at stability and responsive bandwidth as well as cohesiveness between various variables.

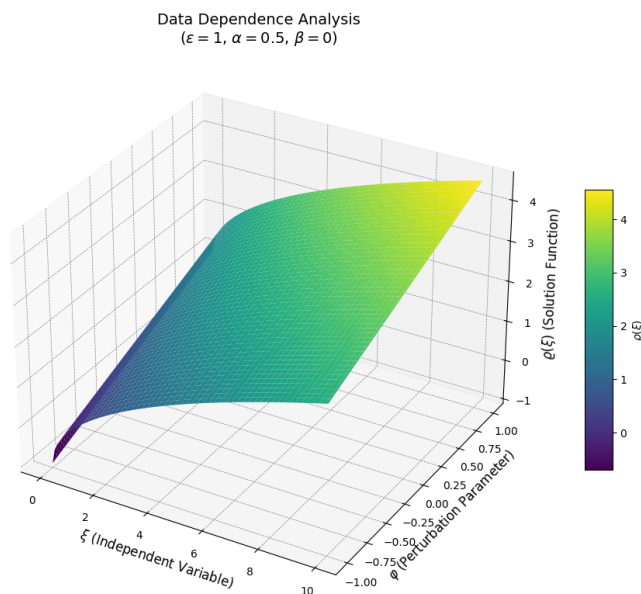


FIGURE 2. 3D surface plot of $\varrho(\xi)$ as a function of ξ and a second parameter φ , with $\beta = 0$, $\epsilon = 1$, and $\alpha = 0.5$.

4. CONCLUSION

The present work has established four different types of Ulam stability of fractional integro-differential equations, applying the Henry-Gronwall inequality, with the underlying functional setting changing in each case. Thereafter, data dependence results were proved through another form of a Gronwall-type inequality involving mixed integral terms. The theory was then complemented by a grossly illustrative example with graphs of the function $\varrho(\xi)$ plotted against some parameters associated with the example, in order to afford meaningful theoretical insights into the results obtained. Nonetheless, there are limitations to the present work. The analysis has been confined to the fractional differential equations involving only one type of fractional derivative and does not embrace systems containing delay terms, stochastic terms, or else multi-term fractional operators. Besides, the graphical simulations have been conditioned on a given set of parameters for which the dynamic content is not explored from the standpoint of more general initial conditions or allowable perturbations. Future research can target the gaps identified by studying systems governed by more general or hybrid fractional derivatives (say Caputo–Hadamard or Hilfer), with added delay and memory kernels, and models arising from physical, biological, or engineering applications. Intensified study of sensitivity of solutions to fractional order changes and their relations to stability properties from different types of integral kernels may also contribute. Thus, higher-dimensional numerical simulations and designing appropriate algorithms that satisfy the computationally efficient implementations of such equations would be precious objects from which to continue searching. This paper lays the groundwork necessary for more advanced and applied studies into fractional integro-differential equations.

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Dr. Raghavendran Prabakaran is an Assistant Professor in the Department of Mathematics at Easwari Engineering College, Chennai, and is currently pursuing his postdoctoral research at Harran University, Türkiye. He earned his Ph.D. in 2025 from Vel Tech University, specializing in fractional differential equations, control theory, and artificial intelligence applications. He has published over 74 research articles in SCIE and Scopus-indexed journals and holds 13 patents. His research interests include mathematical modeling, neural networks, and cryptography. He has received several prestigious recognitions, including Best Researcher and Young Scientist Awards, and actively contributes to international research collaborations and academic peer review.
